Tangent Lines via Algebra

First Semester Calculus

A Problem Set for Homework or for Small-Group Class Discussion

Prerequisites:
1. Familiarity with polynomial division.
2. Familiarity with basic limits and the continuity of polynomial functions.
3. Familiarity with the derivative of polynomials and the computation of tangent lines to graphs of polynomial functions.

Items 1 and 2 can arise when finding the derivatives of x^2, x^3, and x^4 from first principles.

Work on limits usually incorporates at least an informal discussion on the continuity of polynomial functions.

Finding the equation of tangent line to a polynomial curve is a standard application of derivatives in a first-semester calculus course.

THE PROBLEM SET

Here is a surprising result.

Let P be a polynomial of degree $n \geq 2$ and a a real number.

In performing polynomial long division, the quantity $\dfrac{P(x)}{(x-a)^2}$ leaves a remainder a polynomial of degree at most 1. That is, the remainder is an expression of the form $mx+b$, for some real numbers m and b.

Surprisingly

$y = mx + b$ is the equation of the tangent line to the graph of the polynomial at $x = a$.

To get a feel for this claim, try these two problems.

Question 1: *Let P be the polynomial given by $P(x) = x^3 - 2x + 5$.*

a) Use calculus to find the equation of the tangent line to the polynomial at $x = 1$.

b) Compute $\dfrac{P(x)}{(x-1)^2}$ and verify that the remainder term matches your answer to part a).

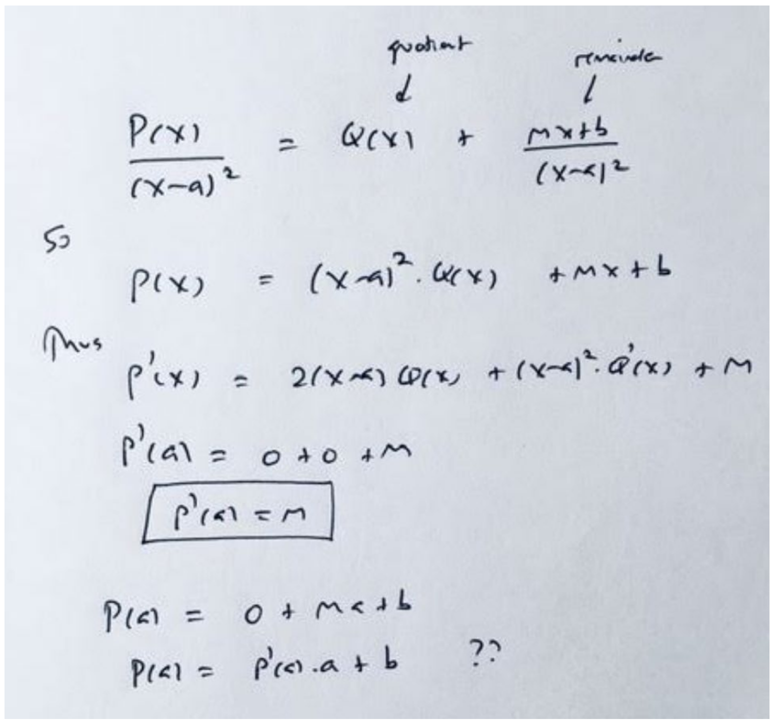
Question 2: *Let* R *be given by* $R(x) = x^2 - 6x + 9 = (x-3)^2$.

a) *What is the equation of the tangent line to the polynomial at* $x = 3$. *Does this match the claim of the theorem?*

b) *What is the equation the tangent line to* R *at a general point* $x = a$ *? Does this match the remainder that arises in dividing* $R(x)$ *by* $(x-a)^2$ *?*

Results of this type allowed scholars of the 1600s to compute tangent lines to curves drawn in the plane without calculus.

Question 3: *Aba wanted to prove the result for herself, even if it meant using calculus. She wrote the following.*



a) *Look at Aba's work. Explain how Abas's first line is a correct interpretaion of the opening comments in the "surprising result" about polynomial division. (Perhaps comment on the nature of the expression represented as* $Q(x)$*.)*

b) *All of the steps following Aba's first line are correct. For example, she correctly applies the product rule in line 3, and her final line with question marks is correct too! What might Aba be trying to get in her final two lines of work? If you have any guesses on this, what might you suggest to Aba to help her thinking?*

c) *Aba says to you that she has a general misgiving about her work. She correctly realises that her first line, upon which all of her reasoning is based, is not meaningful for the value* $x = a$. *Why isn't it?*

d) *Where, later in her work, did Aba implicitly assume using* $x = a$ *was meaningful?*

It is indeed the case that if P is a polynomial of degree $n \geq 2$ and a is a real number, then we can write

$$\frac{P(x)}{(x-a)^2} = Q(x) + \frac{mx+b}{(x-a)^2} \qquad (1)$$

for Q a polynomial of degree $n-2$ and m and b real numbers with this equation valid for any value $x \neq a$.

And it does follow that

$$P(x) = (x-a)^2 Q(x) + mx + b \text{ for all } x \neq a. \qquad (2)$$

To be clear, equations (1) and (2) are valid for all values of x different from a. This begs the question:

Does the equation (2) happen to be valid too for $x = a$?

We can use a limit argument.

It is understood that polynomial functions are continuous. This means that we have

$$\lim_{x \to a} P(x) = P(a)$$
$$\lim_{x \to a} Q(x) = Q(a)$$
$$\lim_{x \to a} mx + b = ma + b$$

Question 4: *Use this to resolve all the worries in questions 3c) and d).*

But Aba is not fully satisfied with this resoution. After all, scholars somehow established this result *without* the use of calculus. She reasoned that maybe scholars of the time had an intuitive sense of limits and continuity and could have argued that the equation

$$P(x) = (x-a)^2 Q(x) + mx + b \qquad (3)$$

is valid for all values of x, even for $x = a$. Let's go with that.

But the tough queston is: What could then lead someone to concluse from equation (3) that $y = mx + b$ is the tangent line to the graph of $y = P(x)$ at the point $(a, P(a))$, and do so without differentiating this equation?

Question 5: *In our scenario, P is a polynomial of degree n and Q a polynomial of degree $n-2$ for some value of $n \geq 2$.*

a) *Why can we be sure that* Q *a polynomial not identically zero?*

A property of continuity that one might want to employ is the following:

There exists an interval $[-\varepsilon,\varepsilon]$ *for some* $\varepsilon > 0$ *such that* $Q(a+h) \neq 0$ *for any non-zero value* h *in this interval.*

b) *Does this property make intuitive sense to you? Does it seem to be believable even if* $Q(a) = 0$ *?*

c) *Use this property to argue that the graphs of* $y = mx + b$ *and* $y = P(x)$ *intersect at* $x = a$ *but not for any value* $x = a + h$ *for a non-zero value* h *in* $[-\varepsilon,\varepsilon]$.

d) *Argue that the line given by* $y = mx + b$ *is indeed the tangent line to the graph of* $y = P(x)$ *at* $x = a$.

LOOKING BACK

Highschool students are asked to divide by polynomials by linear terms and, in many curricula, are asked to explore *The Remainder Theorem* with its conseqent *Factor Theorem*.

If a polymomial P of degree $n \geq 1$ is divided by a linear term $x - a$ for some real number a, then

$$\frac{P(x)}{x-a} = Q(x) + \frac{d}{x-a}$$

where Q is a polynomial of degree $n-1$ and d is the real number $P(a)$.

Consequently, if $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

Question 7: *Practice this. Divide* $P(x) = x^4 + 2x^3 - 3x^2 + 7x - 2$ *by* $x - 1$ *and show that it does indeed leave a remainder of* $P(1) = 5$.

Question 8: *Lu, a high-school student, was shown the following justfication of the Remainder Theorem.*

Computing the division gives

$$\frac{P(x)}{x-a} = Q(x) + \frac{d}{x-a}$$

for some number d .

Multiply through by $x-a$ to see

$$P(x) = (x-a)Q(x) + d .$$

Putting in $x = a$ shows that $d = P(a)$.

Lu had the same misgivings Aba had. They raised their hand and asked: "Why can you just put in $x = a$ *? The very first equation doesn't make sense for* $x = a$ *. Why are we suddenly allowed to put in this value later on?"*

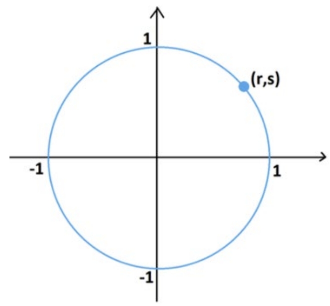
Lu, and their colleagues, have not yet taken any calculus and certainly have not thought about limits and continuity (certainly not in any formal way).

Is there a way to present and make sense of the Remainder Theorem using only the tools and thinking of high-school algebra? How could you allay Lu's concern?

OPTIONAL BONUS CHALLENGES

If you enjoyed the approach of 16^{th}- and 17^{th}-scholars to find equations of tangent lines via purely algebraic means, you might enjoy these questions too.

Bonus 1: Any point (x, y) on the unit circle satisfies $x^2 + y^2 - 1 = 0$.



Let (r, s) be a specific point on the circle and rewrite the defining equation as

$$(x-r)^2 + (y-s)^2 + 2rx + 2sy - 2 = 0.$$

a) Verify that this second equation is indeed equivalent to the equation $x^2 + y^2 - 1 = 0$.

The expression $(x-r)^2 + (y-s)^2$ is strictly positive for all values with either $x \neq r$ or $y \neq s$, and is zero for $x = r$, $y = s$.

b) What is the geometric significance of the line $2rx + 2sy - 2 = 0$?

Bonus 2: More generally, suppose the graph of an equation $A(x, y) = 0$ passes through the point (r, s). What interesting thing can you say about the graph of the equation $(x-r)^2 + (y-s)^2 + A(x, y) = 0$?