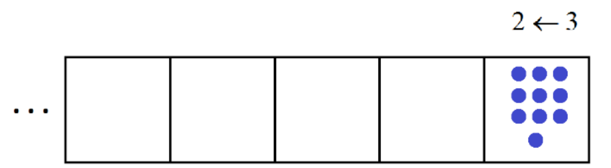
For the full Exploding Dots story (including all the work below) see [here](#).

# The Curious $2 \leftarrow 3$ Machine

## In a $2 \leftarrow 3$ machine, **three** dots in any one box disappear – EXPLODE!—to be replaced with **two** dots just to their left.

For example, placing ten dots in the machine …

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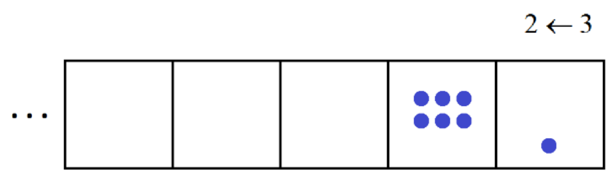


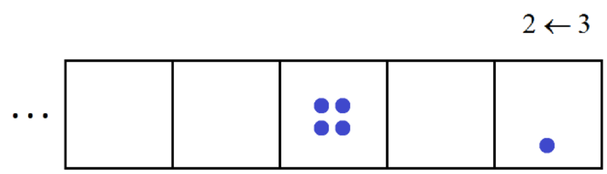


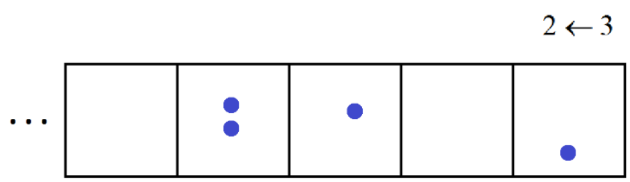


… yields the code $2101$ for ten.

Here are the codes of the first ten numbers.

| | | |
|---|---|---|
| **1: 1** | **6: 210** | **11: 2102** |
| **2: 2** | **7: 211** | **12: 2120** |
| **3: 20** | **8: 212** | **13: 2121** |
| **4: 21** | **9: 2100** | **14: 2122** |
| **5: 22** | **10: 2101** | **15: 21010** |

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## Some Questions

### 1. Is a $2 \leftarrow 3$ machine a place-value machine? If so, what base are we in?

We saw that the code for six is $210$, the code for one is $1$, and the code for thirteen is $2121$. We also know that $6+6+1=13$ in ordinary arithmetic. But look at this:

$$\begin{array}{r} 2\;1\;0 \\ 2\;1\;0 \\ +\qquad 1 \\ \hline = \cancel{4}\;2\;1 = 2121 \end{array}$$

We see six + six + one is thirteen in $2 \leftarrow 3$ machine arithmetic. We can do arithmetic in this machine!

So then, is the $2 \leftarrow 3$ machine just another place-value machine in some weird base? What base?

### 2. Patterns in the codes?

Here are the codes of the numbers zero through forty in a $2 \leftarrow 3$ machine.

| | | | |
|---|---|---|---|
| 0 | | | |
| 1 | 2102 | 21220 | 212021 |
| 2 | 2120 | 21221 | 212022 |
| 20 | 2121 | 21222 | 212210 |
| 21 | 2122 | 210110 | 212211 |
| 22 | 21010 | 210111 | 212212 |
| 210 | 21011 | 210112 | 2101100 |
| 211 | 21012 | 212000 | 2101101 |
| 212 | 21200 | 212001 | 2101102 |
| 2100 | 21201 | 212002 | 2101120 |
| 2101 | 21202 | 212020 | 2101121 |

Are there any patterns to be discovered?

For the full Exploding Dots story (including all the work below) see here.

Here are some specific pattern questions to explore:

a) After a little starting hiccup, do all the codes begin with a 2?
After a slightly bigger starting hiccup, do all the codes being 21?
After a bigger hiccup still, do all the codes begin with the same three digits?

b) There are some codes that end with a single 1.
There are some codes that end 11.
There are some codes that end 111.
Is there a code that ends 1111, or with 11111, or with fifty-three ones?

c) Does it make sense that the final digits cycle 0, 1, 2, 0 , 1, 2, 0, 1, 2, ….?

This gives a divisibility rule for three.

A number is divisible by three only if its final digit (in $2 \leftarrow 3$ machine code) is 0.

Is there a divisibility rule for 9? For 27?

**UNSOLVED PROBLEM:** Is there a natural and easy divisibility rule for 2?
(How do you tell whether or not a number is even just by looking at its code?)

d) Here's something weird:

*Look at the coded of all the even numbers.*
*Delete their last digits.*
*What do notice?*

| | | | |
|---|---|---|---|
| **~~0~~** | | | |
| **~~2~~** | **212~~0~~** | **2122~~1~~** | **21202~~2~~** |
| **2~~1~~** | **212~~2~~** | **21011~~0~~** | **21221~~1~~** |
| **21~~0~~** | **2101~~1~~** | **21011~~2~~** | **210110~~0~~** |
| **21~~2~~** | **2120~~0~~** | **21200~~1~~** | **210110~~2~~** |
| **210~~1~~** | **2120~~2~~** | **21202~~0~~** | **210112~~1~~** |

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## 3. Does the order in which one conducts explosions matter?

Suppose I place ten dots in a machine and do just two explosions in the first box, and then an explosion in the second box, then back to an explosion in the first box, and so on. Do all possible ways I could conduct explosions give the same final $2101$ code for ten?

What if you played with 25 dots in the machine? Does it matter in which order you choose to conduct explosions?

Could the number 25 (or any other for that matter) have two different codes in a $2 \leftarrow 3$ machine?

## 4. Palindromes

The code for five is $22$ and the code of seventeen is $21012$. These are both *palindromes*: they look the same when read forwards and backwards.

Only ten numbers are known to have palindromic $2 \leftarrow 3$ codes. Is there an eleventh one?