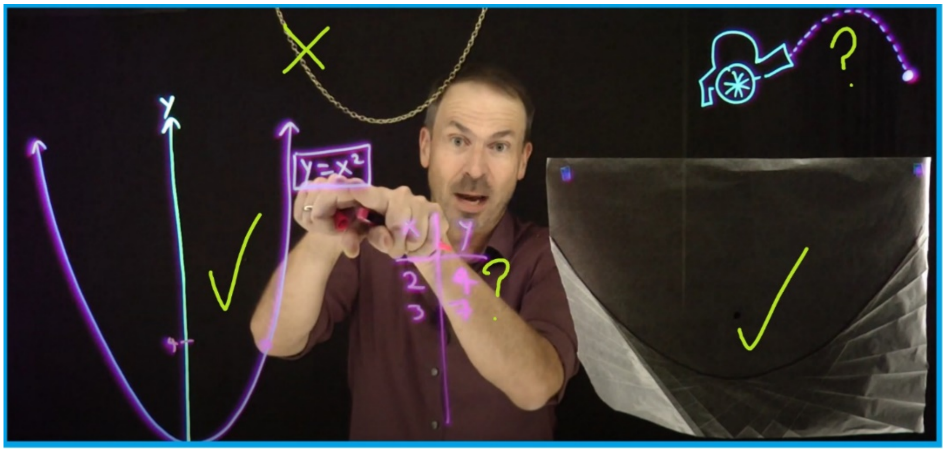
The Power of Symmetry



QUADRATICS

ABOUT THE COURSE

This course is about the beauty and the thinking behind the quadratics: how to think about them and do them -- both their algebra and their graphing -- with natural ease and abundant joy. How? By being absolutely true to their story!

An exploration of quadratics is really an exploration of the power of symmetry, of seeing how this elegant idea so profoundly unites and simplifies. Symmetry is our friend! There really is no need to memorize anything once you realize this core feature of the topic.

This swift course will cover the standard high-school work on quadratics, but do so in a way that teaches how to see through mathematical clutter and to think like a mathematician.

Reference: Written notes for all this content is here.

THE LESSONS

Lessons are subject to the beautiful give-and-take of conversation and so the topics covered in any individual period described here may sway to some degree.

There are 69 practice problems for the course (too many!) and solutions to them appear as the second half of this document.

Lesson 1: **Setting the Scene: What to do if you trust patterns**

Okay: What's the next number 2 4 6 8 __ ?
If you trust pattens you'd likely say 10.

Let's start off the course by playing the game of trusting patterns and seeing what mathematics we can develop. We'll also see how Galileo was hoping to use a pattern to verify a theory of gravitation.

Try practice **Problems 1 through 4.**

Videos to watch if you like: Check out lesson 3.3 here.

Lesson 2: **Still trusting patterns?**

Let's keep playing the game of trusting patterns. But some dubious examples might start creeping in.

Try practice **Problems 5 through 8.**

Videos to watch if you like: Lesson 3.3 from here is still relevant (though feel free to start looking at other lessons in that section too).

Lesson 3: **The play of Area**

Sometimes schoolyards are in the shapes of rectangles and they are called quadrangles. Notice the prefix "quad."

While we're in the mood for play, let's start playing with quadrangles to see how our beliefs about area are intimately connected to arithmetic and algebra. (Have you ever wondered why negative times negative is positive?)

Try practice **Problems 9 through 14.**

Videos: The first part of this video covers this lesson. In fact, the text you see under this video matches the lesson too!

Lesson 4: **The name "quadratic" and the Quadrus Method**

Symmetry is our friend! Let's start playing with symmetrical quadrangles and finally start this course on quadratics! Let's solve quadratic equations. (Why not?)

Try practice **Problems 15 through 18.**

Videos: Check out the videos here and here and here if you want.

Lesson 5: **Mastering the Quadrus Method**

Let's become masters at solving every quadratic equation possible!
(Oh. If you've heard of something called the "quadratic formula" we can discuss that too!)

Try practice **Problems 19 through 28.** Well … don't! That's way too much work. So just pick-and-choose from the problems you see there and do enough until you feel like you really "get it."

Videos: The same videos here and here and here are still relevant.

Lesson 6: **What is graphing?**

People say that "math is a language." What does that actually mean?
Let's chat about that. And let's also figure out what it means to "graph an equation." (It's tied to our answer about language.)

Try practice **Problems 29, 30, 31**.

Videos: The video here covers the content of this lesson.

Lesson 7: **Balancing a Quadratic on your head**

Let's start graphing quadratics. Can we get a graph to balance on my head?

Try practice **Problems 32 - 46**. Again, these are way too many. So, pick and choose.

Videos: The video here covers the content of this lesson.

Lesson 8: **A factor of steepness**

Ooh! Our equations have been "too nice." Let's start graphing more complication quadratic equations now.

Keep going with practice **Problems 32 - 46**. Again, these are way too many. Pick and choose some more!

Videos: The video here covers the content of this lesson.

Lesson 9: **The previous too lessons were too hard. Ignore them!**

We've lost sight of symmetry. We need to bring back symmetry thinking!
*Doing so sets us up for a ridiculously easy way to graph *all* quadratic equations just by using common sense.*

Try practice **Problems 47 - 65**. Again, way too many. So, pick and choose.

Videos: Look at the videos here, here, here, and here if you want. (That's a lot of videos!)

Lesson 10: **How to spell your name in math**

Let's wrap things up. We've basically mastered it all now. So, let's just have some fun writing absurd equations that spell our names!

Videos: Have a look at the website https://globalmathproject.org/personal-polynomial/ and have fun! There are videos there to watch if you want.

ADDITIONAL MATERIAL

If we have time, we will cover the topic of **FACTORING** throughout some of the lessons.

This topic is not actually part of the quadratics story, yet many textbooks insist on covering this work in a quadratics unit. For completeness, we'll try to do the same.

This content is covered here, starting on page 44.

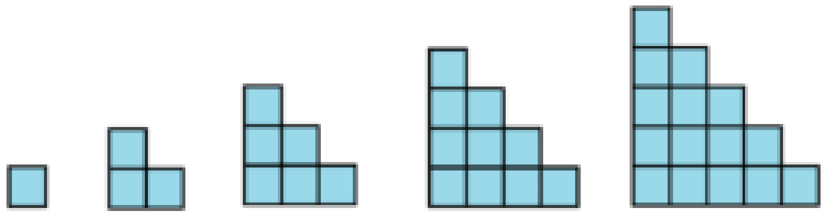
Four practice questions A, B, C, and D appear at the end of the practice problem set.

PRACTICE PROBLEMS

Practice 1: Make an intelligent guess as to the next number in the following sequence.

2 3 6 11 18 27 38 __

Practice 2: Consider the following sequence of diagrams each made of squares 1 unit wide.



If the implied geometric pattern of these first five figures continues …

a) What would be the perimeter of the tenth figure?
b) What would be the area of the tenth figure?

Practice 3: a) Show that for the following sequence it seems that the third differences are constant. Make a prediction for the next number in the sequence.

0 2 20 72 176 350 612 …

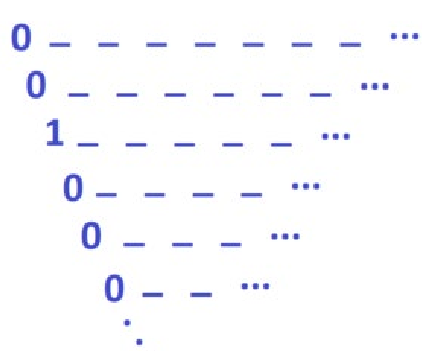
b) Use differences to make an intelligent guess as to the next number in this sequence.

-1 4 7 8 7 4 -1 __

c) How many difference rows must one complete in the sequence below (the powers of two) to see a row of constant differences?

1 2 4 8 16 32 64 128 …

Practice 4: What sequence has **0 0 1 0 0 0 …** as its leading diagonal?



Practice 5: a) Use difference methods to find a formula for the sequence of numbers

2, 2, 4, 8, 14, 22, 32,…

(Just so you have it, the answer is $n^2 - 3n + 4$. Can you see how to get this answer by looking at the leading diagonal?)

b) Use difference methods to show that **0, 2, 10, 30, 68, 130, 222, …** follows the formula $n^3 - 3n^2 + 4n - 1$.

Practice 6: Find formulas for as many of these sequences as you feel like doing.

5, 8, 11, 14, 17, 20, 23, …

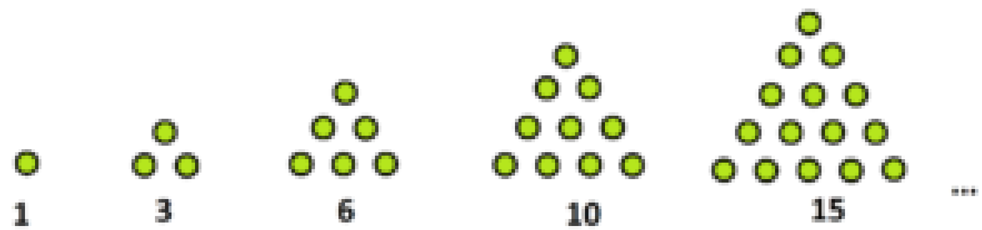
3, 3, 3, 3, 3, 3, 3, 3, 3, …

1, 3, 15, 43, 93, 171, 283, …

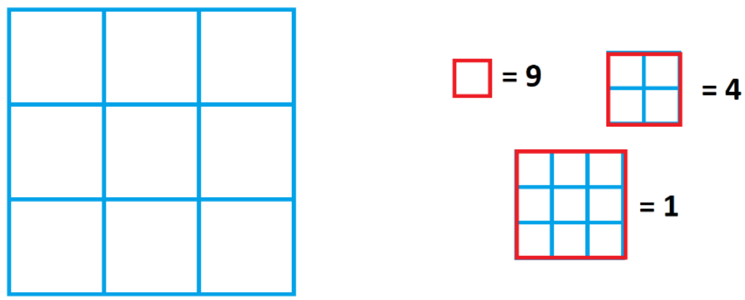
1, 0, 1, 10, 33, 76, 145, 246, 385, …

3, 3, 7, 21, 51, 103, 183, 297, ….

Practice 7: Find a formula for the nth triangular number: 1, 3, 6, 10, 15, 21, 28, 36, ….
(Don't be afraid of fractions!)



Optional Practice 8: Let $S(n)$ be the total number of squares, of any size, one can find in an $n \times n$ grid of squares. For example, $S(3)=14$ because in a three-by-three grid of squares one can find nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of $9+4+1=14$ squares in the grid.



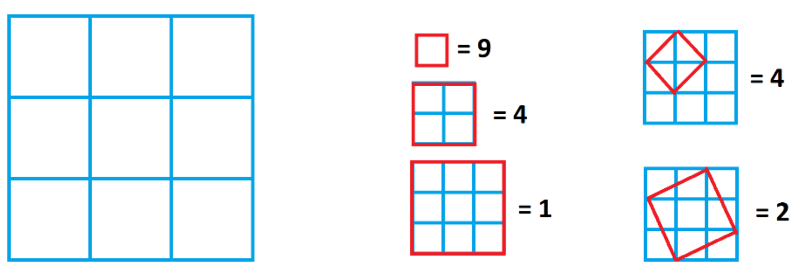
a) Find $S(1)$, $S(2)$, $S(4)$, and $S(5)$.

b) What do difference methods suggest is the general formula for $S(n)$?

c) What is the value of $1^2+2^2+3^2+\cdots+99^2+100^2$?

OPTIONAL:

d) Care to count both tilted and non-tilted squares? For example, there are a total of 20 tilted and non-tilted squares to be drawn on a three-by-three grid.



Practice 9: Use the area model to compute 3721×223.
(Into how many pieces might you divide your rectangle?)

Practice 10: Use the traditional long multiplication algorithm to compute 845×387. And use it again to compute 387×845, but as you do so this second time ask yourself: Is it obvious the algorithm will give the same final answer?

Practice 11: Use the area model to compute $4\frac{1}{3} \times 10\frac{2}{5}$.

Practice 12: Compute 16×15 four different ways to conclude that $(-4) \times (-5)$ is positive 20.

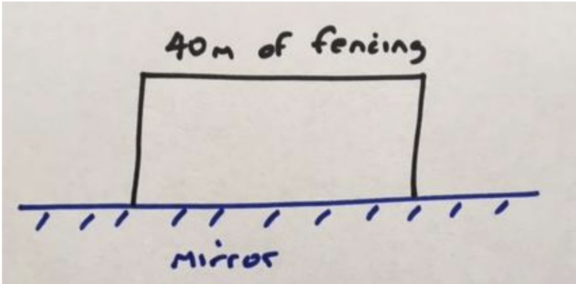
Practice 13:

a) Use the area model to compute $(2x^2 + x + 1)(3x + 2)$.

b) Use your answer to quickly see the value of 211×32.

c) Put $x = -10$ into your answer from a). What multiplication problem is this the answer to?

Practice 14: (CHALLENGE) A farmer has 40 meters of fencing and wants to use it all to make a rectangular pen. But she has huge mirrored wall in her field and wants to use the mirror as one side of her rectangular pen.



What should the dimensions of the rectangle be in order to obtain a pen of maximal area?

Practice 15: Solve

a) $2x^2 = 50$
b) $y^2 + 5 = 14$
c) $100x^2 = 1$
d) $4p^2 = 0.25$
e) $a^2 + 7 = 7$
f) $x^2 = 20$
g) $x^2 = -20$

Practice 16: Solve

a) $(4x-6)^2 = -7$
b) $(4x-6)^2 = 0$
c) $(4x-6)^2 = 4$
d) $(4x-6)^2 = 5$

Practice 17: Solve

a) $(y+1)^2 - 2 = 23$
b) $4(p-2)^2 - 16 = 0$
c) $9 + \left(34x - 77\frac{1}{2}\right)^2 = 0$
d) $\left(x - \sqrt{2}\right)^2 = 5$

Practice 18: Solve

a) $p^2 - 6p + 9 = 9$
b) $x^2 - 4x + 4 = 1$
c) $x^2 - 20x + 100 = 7$
d) $r^2 - 16r + 64 = -2$
e) $x^2 + 2\sqrt{5}x + 5 = 36$
f) $x^2 - 2\sqrt{2}x + 2 = 19$

Practice 19: Solve for x giving your answer in terms of A and B.

$$x^2 + 2Ax + A^2 = B^2$$

Practice 20: Solve

a) $f^2 + 8f + 15 = 80$
b) $w^2 + 90 = 22w - 31$
c) $x^2 - 6x = 3$.

Practice 21: Solve as many of these as you feel like doing.

a) $w^2 - 5w + 6 = 2$
b) $x^2 + 9x + 1 = 11$
c) $p^2 + p + 1 = 0.75$
d) $x^2 = 10 - 3x$
e) $x^2 - x - 1 = 2\frac{3}{4}$
f) $x^2 + 3 = 9$

Practice 22: Solve as many of these you feel like doing.

a) $2x^2 = 9$
b) $4 - 3x^2 = 2 - x$
c) $\alpha^2 - \alpha + 1 = \frac{7}{4}$
d) $3x^2 + 3x + 1 = 19$
e) $-3x^2 + 3x + 1 = 19$
f) $10k^2 = 1 + 10k$

Practice 23: Consider $4x^2 + 6x + 3 = 1$. Does it look like this quadratic equation will have problems when solving it? Does it have problems as you try to solve it? What can you do to obviate the difficulties you encounter?

Practice 24:

a) Design a quadratic equation that has two negative solutions.

b) Design a quadratic equation with just one solution, namely, $x = 4$.

c) Design a quadratic equation with $x = 2$ and $x = 10$ as solutions.

Practice 25:

a) A rectangle is twice as long as it is wide. Its area is 30 square meters. What are the dimensions of the rectangle?
b) A rectangle has one side 4 meters longer than the other. Its area is 30 square meters. What are the dimensions of the rectangle?

Practice 26: Consider $y = 2(x-4)^2 + 6$. What value for x produces the smallest possible value for y? Why?

Practice 27: Find one solution to $(x+1)^3 = 27$.

Practice 28 (OPTIONAL): This problem will require you to multiplying through by 4 many times!

a) Solve $x^2 + x = 2$.
b) Solve $2x^2 + x = 3$.
c) Solve $4x^2 + x = 5$.
d) Solve $8x^2 + x = 9$.
e) Solve $16x^2 + x = 17$.

If you are game …

f) Find the solutions to $2^N x^2 + x = 2^N + 1$.

Practice 29:

a) Sketch a graph of the one-variable equation $x^2 = 4$. (So it's graph will require only one number line, one for x values.)

b) Sketch a graph of the one-variable inequality $x^2 \geq 4$.

Practice 30: Sketch a graph of the two-variable equation $x^2 = y^2$.

Practice 31:

a) Sketch a graph of the one-variable equation $x = 3$.

b) Sketch a graph of $x = 3$ thinking of it as a two-variable equation. (Imagine it as $x + 0 \cdot y = 3$ if you like.)

c) Sketch a graph of $x = 3$ thinking of it as a three-variable equation. (Imagine it as $x + 0 \cdot y + 0 \cdot z = 3$ if you like.) How will you draw your three number lines?

Practice 32:

a) Sketch the graph of $y = (x-10)^2$.

b) Sketch the graph of $y = (x+5)^2$.

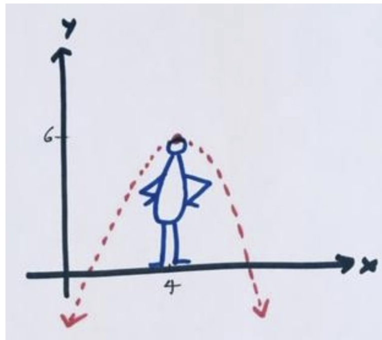
Practice 33:

a) Sketch the graph of $y = (x-5)^2 + 2$.

b) Sketch the graph of $y = (x+5)^2 - 2$.

Practice 34: When we say that the graph of $y = x^2$ is a U-shaped graph, is that a correct analogy? The two sides of the letter "U" are vertical. Does the graph of $y = x^2$ possess vertical lines?

Practice 35: Find three different equations that give U-shaped graphs that balance on my head this way.



Practice 36: Draw, on the same sets of axes, rough sketches of each the following equations.

$y = x^2$ $\quad y = 1.1x^2$ $\quad y = 0.9x^2$

$y = -x^2$ $\quad y = -1.1x^2$ $\quad y = -0.9x^2$

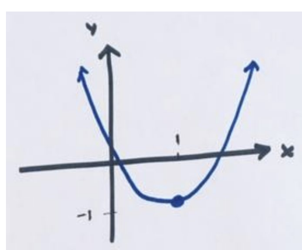
Practice 37: Sketch graphs of

a) $y = 3(x-5)^2$

b) $y = 3(x-5)^2 + 4$

c) $y = -2(x+4)^2 + 40$.

Practice 38: Which of the following equations could have the graph shown?



a) $y = \frac{4}{3}(x+1)^2 + 1$

b) $y = -\frac{4}{3}(x+1)^2 + 1$

c) $y = \frac{4}{3}(x-1)^2 + 1$

d) $y = -\frac{4}{3}(x-1)^2 + 1$

e) $y=\frac{4}{3}(x+1)^2-1$

f) $y=-\frac{4}{3}(x+1)^2-1$

g) $y=\frac{4}{3}(x-1)^2-1$

h) $y=-\frac{4}{3}(x-1)^2-1$

Practice 39: Sketch the graph of $y=-2(x+10)^2-7$.

Practice 40: The graph of a quadratic equation has a vertical line of symmetry at $x=3$, and has highest value $y=17$. Which of the following could be an equation for that quadratic?

a) $y=200(x-3)^2+17$

b) $y=-200(x-3)^2+17$

c) $y=200(x-3)^2-17$

d) $y=-200(x-3)^2-17$

Practice 41: Sketch a graph for each of the following equations.

a) $y=2-x^2$

b) $y=\frac{1}{3}\left(x-\frac{1}{2}\right)^2-4$

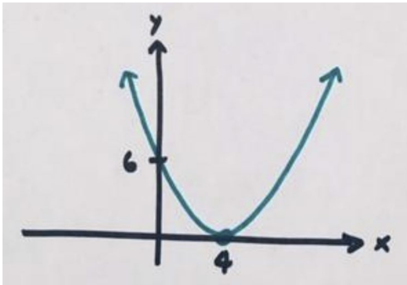
c) $y=0.0034(x+0.276)^2+0.778$

d) $y=200000(x-200000)^2-200000$

Practice 42: If $y=a(x+b)^2+c$ has a graph passing through the origin and with $(2,3)$ as the vertex, then what is the value of $a+b+c$?

a) $\frac{1}{4}$ b) $1\frac{3}{4}$ c) $4\frac{1}{4}$ d) $5\frac{1}{4}$

Practice 43: Write a quadratic equation that fits this graph.



Practice 44: Write down a quadratic equation whose graph passes through the points $(3,18)$ and $(17,18)$ and has lowest value 5.

Practice 45: Write down a quadratic equation whose graph passes through the x axis at $x=-2$ and at $x=10$ and passes through the y axis at $y=-6$.

Practice 46: Write down quadratic equations with symmetrical U-shaped graphs possessing the following properties:

a) Crosses the x-axis at 3 and 5 and the y-axis at 1000.

b) Passes through $(4,10)$, $(6,10)$ and $(8,13)$.

c) Has vertex $(5,5)$ and passes through $(4,4)$.

d) Has vertex the origin and passes through the point $(\sqrt{2},\pi)$.

Practice 47: The graph of a quadratic equation passes through the points $(3,81)$, $(4,9)$, and $(-10,9)$. What is the x-coordinate of its vertex?

Practice 48: Sketch a graph of $y=2(x-3)(x-23)+200$.

Practice 49: Sketch a graph of $y=-(x-3)(x+5)+6$.

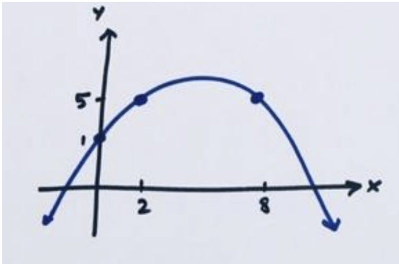
Practice 50: Sketch a graph of $y=-2x(x-80)+3$.

Practice 51: Which value of r forces the graph of $y=3x^2+6x+r$ have 5 as the smallest possible y value?

Practice 52: Find a negative value for a so that $y=x^2+ax+a$ has smallest possible value -3.

Practice 53: Find a formula for the location of the line of symmetry of a general quadratic equation $y=ax^2+bx+c$.

Practice 54: Find the quadratic equation whose graph appears as shown.



Practice 55: Write down a quadratic equation whose graph has x intercepts $x=-3$ and $x=11$ and y intercept 10.

Practice 56: Sketch the graph of $y=-3x^2-18x+5$ and then use the graph to rewrite the equation in "vertex form."

Practice 57: Consider the equation $y=3x^2-6x+20$.

a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x intercepts of the graph.
f) Find the y intercept of the graph.

Practice 58: Consider the equation $y=5x^2-10x$.

a) Find its vertex.
b) Find its axis of symmetry.
c) Rewrite the equation in vertex form.
d) Sketch its graph.
e) Find the x intercepts of the graph.
f) Find the y intercept of the graph.

Practice 59: Solve the following quadratic equations.

a) $(x-3)(x+5)=1$

b) $x^2=(2x-1)(2x+1)-5$

c) $(x-10)(x+1)+5=12$

Practice 60: Find, in terms of c, the value k so that $y=(x+c)(x-c)+k$ gives -2 as the smallest possible y value.

Practice 61:

a) Solve $x^2+10x+30=0$ and see what happens when you try.

b) Sketch the graph of $y=x^2+10x+30$.

c) Use the graph to explain what happened in part a)

d) Will $x^2+10x+30=11$ have a solution? If so, how many solutions?

e) For which value(s) k does $x^2+10x+30=k$ have only one solution?

Practice 62: How many solutions does $-x^2+4x-5=0$ have? Answer this question not by algebra, but by graphing.

Practice 63: a) Find a value k so that the graph of

$$y=5x^2-10x+k$$

just touches the x axis.

b) Find a value m so that $y=-2x^2-18x+m$ gives the highest output value of 100.

c) Find a value p so that $y=(x-p)(x-3p)$ has smallest output value of -10.

Practice 64: A rectangle has side lengths $7-r$ and $3+r$ for some value r. What value for r gives a rectangle of maximal area?

Practice 65: Here are three quadratic equations:

(A) $y=3(x-3)(x+5)$

(B) $y=2x^2+6x+8$

(C) $y=2(x-4)^2+7$

i) For which of these three expressions is it very easy to answer the question: "What is the smallest y-value the expression can produce?"

ii) For which of these three expressions is it very easy to answer the question: "Where does the graph of the equation cross the x-axis?"

iii) For which of these three expressions is it very easy to answer the question: "Where does the graph of the quadratic cross the y axis?"

iv) For which of these three expressions is it very easy to answer the question: "What are the coordinates of the vertex in this equation's graph?"

FACTORING QUESTIONS

Question A: Find the missing factor

a) $\underline{\qquad\qquad}\cdot(x+3)=2x^2+8x+6$

b) $\underline{\qquad\qquad}\cdot(x+5)=x^2+x-20$

Answer: https://youtu.be/r6sNtQsiJSc

Question B: Factor each of the following. (They have each been designed to factor.)

a) $x^2+7x+12$

b) $x^2-7x+12$

c) $x^2+4x-60$

d) $2x^2+5x+2$

Answer: https://youtu.be/q3AmQhVpeWU

Question C: Factor each of the following. (They have each been designed to factor.)

a) x^2-16

b) $9x^2-16$

c) $81-x^2$

d) $-p^2q^2+a^2$

e) $x^2-4+xy+2y$

Answer: https://youtu.be/2mkqEoPeqvk

Question D:

a) Factor $\frac{p^2}{4}-\frac{1}{9}$.

b) Does x^2-5 factor?

c) Show that n^3-8n^2+5n-6 does not factor as $(n^2-2n+3)(n+2)$.

Answer: https://youtu.be/hQZnmpSqnTE