# QUADRATICS
# 3.3 More Practice; The Difference of Two Squares; The Opening Puzzle

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## SETTING THE SCENE

In the previous essay we factorized the expressions

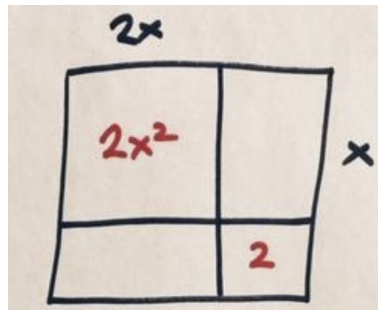
$$x^2+5x+6$$
$$x^2+9x+20$$
$$x^2-8x+12$$
$$x^2-5x+6$$
$$x^2+7x+12$$
$$x^2+2xt-15t^2$$

—and more!—with each expression beginning with $x^2$. But what if the coefficient out front is not $1$? We start this essay with that issue.

## MORE FACTORING

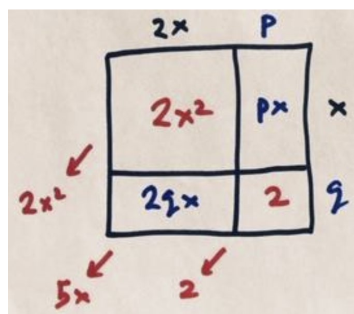
**PROBLEM:** *Factorise* $2x^2+5x+2$.

Can we write this quadratic expression as an (unsymmetrical) rectangle? We see have a piece of area $2x^2$. As a first guess, we might choose to think of this as $2x \times x$.




In which case see we seek two numbers $p$ and $q$ with

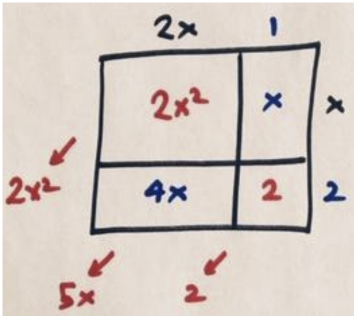
$$p+2q=5$$
$$pq=2.$$




This pair of equations is subtle: we need two factors of $2$ such that one factor plus double the other factor adds to $5$. One eventually comes to think of

$$p=1$$
$$q=2.$$

And we see that this does indeed work!




Thus $2x^2 + 5x + 2$ factors as $(2x+1)(x+2)$.

So even if there is a coefficient different from $1$ associated with the $x^2$ term in a quadratic expression, we can use intelligent guesses and logic to still, perhaps, work things through.

**PRACTICE 1:** *Factorise* $3x^2 - 8x + 4$.

**PRACTICE 2:** *Which of the following expressions can you factorise? (Warning: Only some of them have been carefully crafted to factorise nicely, if at all!)*

*a)* $4x^2 - 11x - 3$
*b)* $-x^2 + 4$
*c)* $10x^2 + 19x - 2$
*d)* $x^2 - 6x + 4$
*e)* $9x^2 - 6x - 4$
*f)* $703x^2 + 141302x - 201$

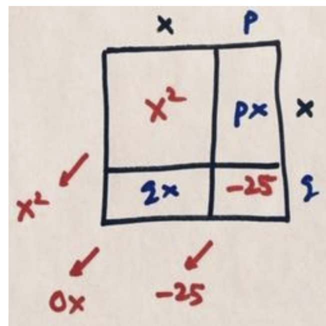
## DIFFERENCE OF TWO SQUARES

**PROBLEM:** *Factorise* $x^2 - 25$.

This quadratic expression looks like it is missing a term. But we can, if you like, think of this as

$$x^2 + 0x - 25.$$

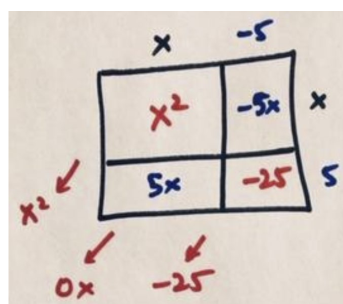
Let's try to represent this as an unsymmetrical rectangle.




We need two numbers $p$ and $q$ satisfying

$$p + q = 0$$
$$pq = -25.$$

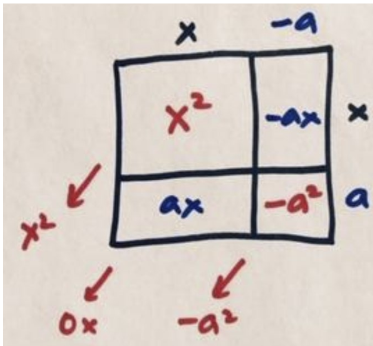
So $p = -q$ and multiply to $-25$. Let's go with $p = 5$ and $q = -5$, and we see it works.




We have

$$x^2 - 25 = (x-5)(x+5).$$

This same work shows we can factorise any expression of the form $x^2 - a^2$ in just the same way.




This gives a famous formula in mathematics called the **difference of two squares** formula.

$$x^2 - a^2 = (x-a)(x+a)$$

This formula shows that any number that is the difference of two square numbers factors as a product of two numbers. (People seem to forget algebra formulas are really formulas about numbers!) For example, the difference of two squares formula gives

$$\begin{aligned} 49 - 16 &= 7^2 - 4^2 \\ &= (7-4)(7+4) \\ &= 3 \times 11, \end{aligned}$$

and the difference of $49$ and $16$ is indeed $33$.

## SOLVING THE OPENING PUZZLE

Recall we were wondering how one could know that the 91-digit number $2^{300} - 1$ is not prime. The answer lies in factoring.

Scholars like Mersenne were very interesting in general factoring formulas, such as the difference of two squares, to answer hard questions in number theory. For instance, if one notices that $2^{300}$ and $1$ are each square numbers (and they are!) then we can show that their difference factors. We have

$$\begin{aligned} 2^{300} - 1 &= \left(2^{150}\right)^2 - (1)^2 \\ &= \left(2^{150} - 1\right)\left(2^{150} + 1\right) \end{aligned}$$

proving that $2^{300} - 1$ is not prime. Whoa!

**PRACTICE 3:**

*a) Use the difference of two squares formula to explain why* $2^{10} - 1$ *is divisible by* $3$, $11$, *and* $31$.

*b) Find two four-digit factors of* $2^{20} - 1$.

**PRACTICE 4:** *Find a five-digit factor of* $99999996$.

**PRACTICE 5:**

*a) Convince me that* $2^{300} - 25$ *is not prime.*

*b) Convince me that* $2^{300} - 26$ *is not prime.*

*c)* **CHALLENGE:** *Convince me that* $2^{300} - 27$ *is not prime.*

# SOLUTIONS

**PRACTICE 1:** *Factorise* $3x^2-8x+4$

**Brief Answer:** This is $(3x-2)(x-2)$.

**PRACTICE 2:** *Which of the following expressions can you factorise? (Warning: Only some of them have been carefully crafted to factorise nicely, if at all!).*

*a)* $4x^2-11x-3$
*b)* $-x^2+4$
*c)* $10x^2+19x-2$
*d)* $x^2-6x+4$
*e)* $9x^2-6x-4$
*f)* $703x^2+141302x-201$

**Brief Answers:**
a) This is $(4x+1)(x-3)$.

b) This is $(-x+2)(x+2)$.

c) This is $(10x-1)(x+2)$.

d) I don't see how to readily factor this one. (It turns out to be $\left(x-3+\sqrt{5}\right)\left(x-3-\sqrt{5}\right)$. And I know this only because I started with this expression to create the question!)

e) I don't see how to readily factor this one. (It turns out to be $\left(3x-1+\sqrt{5}\right)\left(3x-1-\sqrt{5}\right)$. Again, this is what I started with to create the question.)

f) This is $(703x-1)(x+201)$, though these are hard numbers to guess!

**PRACTICE 3:**

*a) Use the difference of two squares formula to explain why* $2^{10}-1$ *is divisible by* $3$, $11$, *and* $31$.

*b) Find two four-digit factors of* $2^{20}-1$.

**Answers:**

a) We have

$$\begin{aligned}2^{10}-1&=\left(2^5\right)^2-(1)^2\\&=32^2-1^2\\&=(32-1)(32+1)\\&=31\times 33\end{aligned}$$

and this equals $3\times 11\times 31$.

b) We have

$$\begin{aligned}2^{20}-1&=\left(2^{10}\right)^2-1^2\\&=1024^2-1^2\\&=1023\times 1025.\end{aligned}$$

**PRACTICE 4:** *Find a five-digit factor of* $99999996$.

**Answer:** This number is $100000000-4$, which is $10^8-4$. And we have

$$\begin{aligned}10^8-4&=\left(10^4\right)^2-2^2\\&=(10000)^2-2^2\\&=9998\times 10002.\end{aligned}$$

So $10002$ is an example of a five-digit factor of the number.

**PRACTICE 5:**

a) *Convince me that* $2^{300}-25$ *is not prime.*
b) *Convince me that* $2^{300}-26$ *is not prime.*
c) **CHALLENGE:** *Convince me that* $2^{300}-27$ *is not prime.*

**Answers:**

a) We have

$$\begin{aligned}2^{300}-25 &= \left(2^{150}\right)^2-5^2\\ &= \left(2^{150}-5\right)\left(2^{150}+5\right)\end{aligned}$$

and so the number factors and is not prime.

b) $2^{300}-26$ is the difference of two even numbers and so is even. It has $2$ as a factor and so is not prime.

c) This one is tricky! We'll answer this question in the next lesson.