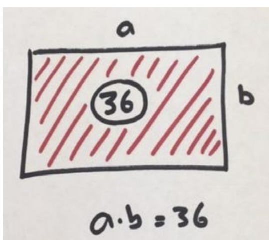
QUADRATICS
3.2 Breaking Symmetry: Factoring

James Tanton

SETTING THE SCENE

Recall that we started our story of symmetry with a rectangle of area 36.

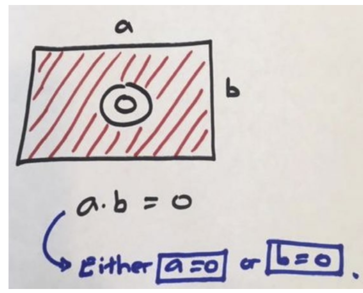


Without any further detail, we can say nothing about the width and length a and b of that rectangle: maybe it's a 4-by-9 or a 2-by-18 or a 45-by-0.8 rectangle. We can't know.

But if we add the word "symmetrical" to our description of the rectangle, then we know something about the values a and b: they must each be 6.

But there is one other instance where one could say something about the side lengths of a given rectangle.

Suppose I gave you a rectangle and told you it had ZERO AREA.



Then you would say that the picture is wrong and that we should have drawn a "rectangle" with either zero width or zero length. That is, we'd deduce that either $a = 0$ or $b = 0$.

People have observed that this one special case, of having a rectangle of zero area, can be used to solve some certain quadratics—if we are lucky!

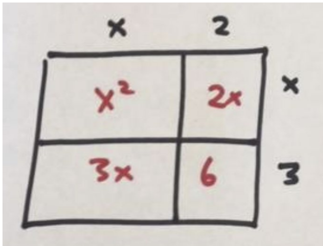
This essay shows how.

UNSYMMETRICAL RECTANGLES OF ZERO AREA

Consider the algebraic expression

$$(x+2)(x+3) .$$

We recognize this as the area of an (unsymmetrical) $x+2$ by $x+3$ rectangle.

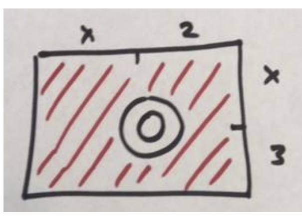


We see that the area equals $x^2 + 5x + 6$.

Now let's reverse this. Suppose I asked you to solve the quadratic equation

$$x^2 + 5x + 6 = 0.$$

We know the left side is the area of an unsymmetrical rectangle, and we're being told that the area of that rectangle is zero.



Well this means that one of the sides of the rectangle must be zero length:

Either $x + 2 = 0$ or $x + 3 = 0$.

We deduce then that either $x = -2$ or $x = -3$ and we have two solutions to the quadratic equation! (Check: Verify that $x^2 + 5x + 6$ does equal zero if x is -2 or if x is -3.)

UPSHOT: If I can recognize a quadratic expression as an (unsymmetrical) rectangle, and if I am being told that the area of that rectangle is zero, then I can solve the equation by noting that one of the side lengths must be zero.

Warning: There are "ifs" in that statement! It is usually not possible to solve a quadratic equation this way and, even if it is possible, one still needs to make guesses (intelligent guesses!) and rely on some luck.

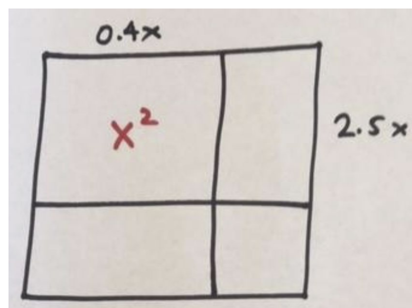
FACTORING

Consider

$$x^2 + 9x + 20 = 0.$$

Can we solve this equation be recognizing the left side as an unsymmetrical rectangle? Might luck be on our side?

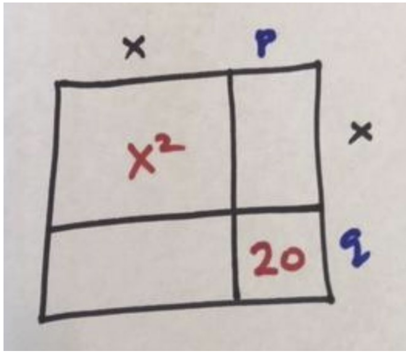
We can start by drawing a picture of a rectangle and noting that there will be an x^2 piece. But because we are no longer using symmetry we can't be sure if this comes from x times x (the symmetrical start) or from $2x$ times $\frac{1}{2}x$ or from $0.4x$ times $2.5x$ or something else!



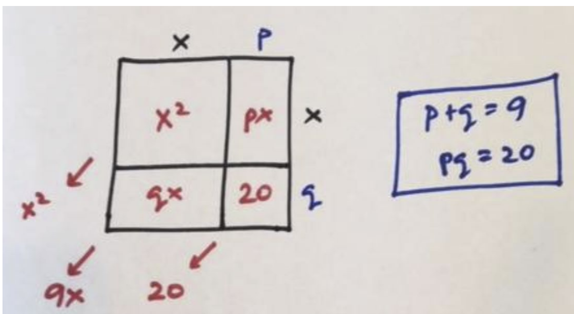
It seems like a natural first guess might be to go with $x^2 = x \times x$. If that doesn't work we can try something else (or better, we can just go back and solve the equation by the proper symmetrical quadrus method!)

We also have a piece of area 20, which could come from 4×5 or from 10×2 or something else. We don't know!

But let's put in some abstract numbers, p and q, and see if we can guess what they might be from the picture. For starters, we see we need $pq = 20$.



Filling in the rectangle we see that we also need $px + qx$ to equal $9x$, that is, we need $p + q = 9$.



So we're looking for two numbers which sum to 9 have product 20.

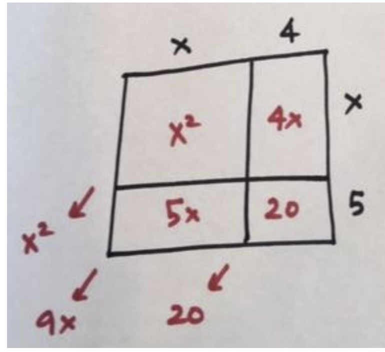
As we saw in last lecture such sum/product puzzles need not have solutions!

But have we lucked out here in this example? Can we think of two factors of 20 that sum to 9. Yes! The numbers 4 and 5 work.

So let's set $p = 4$ and $q = 5$.

(Ooh! Or should it be $p = 5$ and $q = 4$? If things don't work out the first way round we can always come back and try this second order.)

We do see that all falls into place with these numbers.



So we have succeeded in rewriting $x^2 + 9x + 20$ as an (unsymmetrical) rectangle.

Since we want to solve $x^2 + 9x + 20 = 0$, we are being told that this rectangle has zero area. So either $x + 4 = 0$ or $x + 5 = 0$ and we deduce

$$x = -4 \text{ or } -5$$

are solutions. (Double check: Do each of these values indeed make $x^2 + 9x + 20$ equal to zero?)

Do you see that there was a lot of luck involved in this process?

1. We first went with the guess that x^2 came from $x \times x$. And it happened to work.

2. We then had to solve a sum/product puzzle with $p + q$ and pq and it was lucky we could solve it.

3. And it was lucky that we could solve it with nice whole numbers that we could recognize!

4. Then we could work with an (unsymmetrical) rectangle whose area, we were told, is zero,

This process of rewriting a quadratic expression, if one can, as the product of two sides of a rectangle is called **factoring** the quadratic.

People either use the command “factorise” or “factor” to an expression. (It depends on whether you are working with a UK- or an US-based curriculum.)

Success in factoring relies on a lot of luck. As you attempt to factorise a quadratic expression you need to make a host of intelligent guesses and still cross your fingers that everything will work out.

But the truth is that factoring rarely works! Most quadratics won’t factor with nice whole numbers and many won’t factor at all.

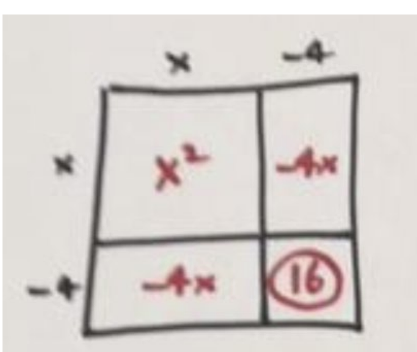
But most school curricula want students to practice the art of making intelligent guesses and pushing through and so will present example after example of quadratics that just, by luck, happen to factor nicely. This often gives students the impression that factoring is a standard and fruitful solving technique and that luck will always be on their side.

Just keep in mind, factoring generally only works for specially designed examples found in textbooks and exams.

PROBLEM: *Solve* $x^2-8x+12=0$ *three different ways. Use the (symmetrical) quadrus method, use the quadratic formula, and use (unsymmetrical) factoring.*

Here goes.

Symmetry: The quadrus method has us work with $x^2-8x+16=4$.



We have

$$(x-4)^2=4$$

$$x-4=2 \text{ or } -2$$

$$x=6 \text{ or } 2.$$

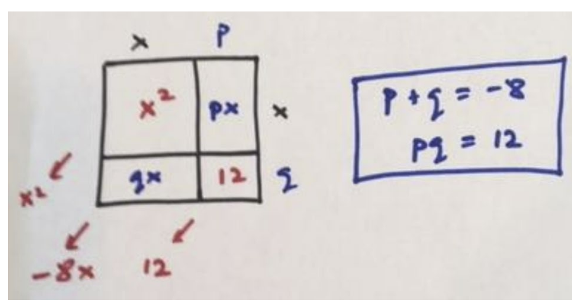
The Quadratic Formula: Fortunately, this equation is already set to equal zero on the right, which is the form of the equation the quadratic formula wants. We have $a=1$, $b=-8$, and $c=12$. We get

$$x=\frac{-(-8)\pm\sqrt{64-4\cdot1\cdot12}}{2\cdot1}$$
$$=\frac{8\pm\sqrt{16}}{2}$$
$$=\frac{8\pm4}{2}$$

so

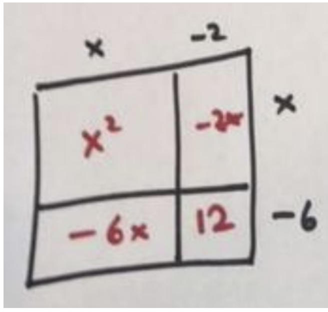
$$x=\frac{12}{2}=6 \text{ or } \frac{4}{2}=2.$$

Non-symmetry: Let’s draw a rectangle and guess that x^2 comes from $x\times x$.



We seek two values p and q with sum -8 and product 12. It takes some mulling, but eventually we think of $p=-2$ and $q=-6$. (Or perhaps it is the other way round?)

We see these values do work!



Since this is a rectangle of zero area, we must have

$x-2=0$, giving $x=2$

or

$x-6=0$, giving $x=6$.

PRACTICE 1: *Solve each of the following by unsymmetrical factoring. (Be sure to get to part d.)*

a) $x^2-5x+6=0$
b) $x^2+7x+12=0$
c) $x^2+2xt-15t^2=0$

(In this one, the coefficients have another variable in them! $a=1$, $b=2t$, $c=-15t^2$. Weird!)

d) $x^2-7x+103=0$

PRACTICE 2: *Solve the following by factoring.*

a) $x^2-9=0$
b) $x^2=x+12$
c) $w^2-35w+\frac{1}{2}=750\frac{1}{2}$

FINAL COMMENT

Did you indeed try part d) of the first practice set? Did you indeed manage to "factorise" it? I know the answer is that you didn't because this quadratic expression $x^2-7x+103$ simply doesn't factor: there are no two numbers with sum -7 and product 103 (even if you entertain working with fractions or irrational numbers).

As I said, this factoring technique rarely works. If you are lucky, and make the right intelligent guesses along the way, you might be able to solve a quadratic equation via this method.

Textbooks like to provide questions where they are certain students can be lucky.

PRACTICE 3: *Attempt to solve* $x^2+4x+2=0$ *by factoring. (This equation does have solutions.)*

OPTIONAL ACTIVITY FOR COMPUTER PROGRAMMERS:

Consider a general quadratic equation of the form $x^2+bx+c=0$. *This will factor with integer values throughout if we can find two whole numbers* p *and* q *satisfying*

$$p+q=b$$
$$pq=c\ .$$

a) Suppose b *and* c *are each one-digit numbers, each from number from the set* -9, -8, …, -1, 0, 1, …, 9. *There are then* 121 *different equations* $x^2+bx+c=0$ *to consider.*

Write a computer program that determines the percentage of these equations that factorise with whole number values.

b) Suppose instead b *and* c *are instead each two-digit numbers. There are now 32,400 different equations* $x^2+bx+c=0$ *to consider.*

Write a computer program that determines the percentage of these equations that factor with whole number values.

SOLUTIONS

PRACTICE 1: *Solve each of the following by unsymmetrical factoring. (Be sure to get to part d.)*

a) $x^2 - 5x + 6 = 0$
b) $x^2 + 7x + 12 = 0$
c) $x^2 + 2xt - 15t^2 = 0$

(In this one, the coefficients have another variable in them! $a = 1$, $b = 2t$, $c = -15t^2$. Weird!)

d) $x^2 - 7x + 103 = 0$

Brief Answers:
a) Drawing the rectangle suggest we need to numbers p and q with

$$p + q = -5$$
$$pq = 6.$$

One thinks of $p = -2$ and $q = -3$, and we see that $x^2 - 5x + 6$ equals $(x-2)(x-3)$. This means we need to solve

$$(x-2)(x-3) = 0.$$

So either $x - 2 = 0$ or $x - 3 = 0$, that is, either $x = 2$ or $x = 3$.

b) One finds that this equation is equivalent to $(x+3)(x+4) = 0$ and so $x = -3$ and $x = -4$ are solutions.

c) Drawing the rectangle suggest we need to numbers p and q with

$$p + q = 2t$$
$$pq = -15t^2.$$

Some mulling suggests $p = 5t$ and $q = -3t$, and this works! Our equation is

$$(x+5t)(x-3t) = 0.$$

This gives $x = -5t$ or $x = 3t$.

d) There are no numbers p and q with

$$p + q = -7$$
$$pq = 103.$$

This quadratic does not factor. (And if you check with the quadrus method or with the quadratic formula, this equation has no solutions.)

PRACTICE 2: *Solve the following by factoring.*

a) $x^2 - 9 = 0$
b) $x^2 = x + 12$
c) $w^2 - 35w + \frac{1}{2} = 750\frac{1}{2}$

Brief Answers:
a) $(x-3)(x+3) = 0$ so $x = 3$ or -3.
b) $x^2 - x - 12$ factors as $(x-4)(x+3)$ and so $x^2 - x - 12 = 0$ has solutions $x = 4$ and $x = -3$.
c) $w^2 - 35w - 750$ factors as $(w-50)(w+15)$ and so $(w-50)(w+15) = 0$ has solutions $w = 50$ and $w = -35$.

PRACTICE 3: *Attempt to solve* $x^2 + 4x + 2 = 0$ *by factoring. (This equation does have solutions.)*

Answer: Did you naturally deduce that $x^2 + 4x + 2$ factors as $\left(x + 2 + \sqrt{2}\right)\left(x + 2 - \sqrt{2}\right)$? Did you think of $p = 2 + \sqrt{2}$ and $q = 2 - \sqrt{2}$ as the two numbers that sum to 4 and have product 2?

This factoring technique is rarely manageable!