TEN-ADIC THINKING WITH EXPLODING DOTS

Part 1: A TROUBLING NUMBER FOR OUR USUAL MATHEMATICS

Here's a deep age-old question regularly asked by young math scholars across the globe:

> *Is $0.9999...$ equal to one or is it not?*

This is a quantity with infinitely many 9s listed to the right of the decimal point.

Well, let's be honest, that's a dubious issue at the outset: we never actually list infinitely many 9s to the right. We humans never can. We might have the patience to write down twenty 9s, or two hundred 9s, or we might hire a team of two thousand people to work for twenty years listing our two billion of those 9s , but we humans will never see, actually see, an infinite number of 9s written to the right of the decimal point. We have to cheat and write an ellipsis, "$\ldots$," to say "imagine this going on forever."

So right away we should realise that the question asked is one for the mind. It is really asking:

> *If we were somehow God-like and could list an infinite string of 9s to the right of the decimal point, would the quantity that results be equal to 1 , or not?*

And one might decide that since we humans are not God-like the question is meaningless—what does "the quantity that results" mean?—and so there is no point to even giving the question consideration.

That's a valid stance to take.

But, nonetheless, this question keeps coming up. We are teased by it. And it seems we do want to try to play with it and make sense of it. Somehow it feels as though we humans, although not able to actually see and write down the infinite, can somehow envision the infinite, that we're just a small step away from having a grasp of it.

It is true that each approximation we humans write for $0.9999...$ is smaller than 1.

> 0.9 is less than 1.
> 0.99 is less than 1.
> 0.99999 is less than 1.
> 0.99999999999999 is less than 1.
> $0.999999999999999999999999999999999999999999999$ is less than 1.

And some use this as the basis for an argument that $0.9999...$ too must be smaller than 1, and so not1.

But let's think a little more deeply about these numbers 0.9 , 0.99 , 0.999 , …. They're are sequence of numbers marching rightward on the number line, heading to the number $0.9999....$, which might or might not be 1.

At the same time, we see the more 9s we list after the decimal point, the closer the numbers are to the number 1. For instance, 0.99 is one-hundredth of a unit away from 1, and 0.999999 is one-millionth of a unit away from 1.

So in fact the numbers 0.9, 0.99, 0.999, … steadily march rightward on the number line eventually entering any amount of space you might specify sitting just to the left of 1. (Want a number within a distance of $\frac{1}{337}$ from 1? 0.9999 will do!)

So could the quantity $0.9999...$ be a number sitting just shy of the number $1$and not be 1? Could there be any empty space between it and 1?

No! We just argued that the sequence of numbers 0.9, 0.99, 0.999, … will march into any space we specify to the left of 1! We conclude:

> *There can be no space between $0.9999\ldots$ and 1 on the number line, and so $0.9999\ldots$ must actually be 1.*

Well … not quite!

There is another possibility to rule out:

> *Could $0.9999....$ instead be quantity sitting just to the right of 1 on the number line?*

It feels unlikely since all the numbers 0.9, 0.99, 0.999, … sit to the left of $1$with 1 being an upper barrier to them all.

But I personally don't see how to argue with the geometry of the number line how to easily rule out this possibility. We need another argument.

An Algebraic Argument

If you choose to believe that $0.9999....$ is a valid arithmetical quantity (that might or might not be 1), then you probably would also choose to believe that it obeys all the usual rules of arithmetic. If so, then we can conduct a lovely swift algebra argument so show that, because of these beliefs, the quantity $0.9999...$ does indeed equal 1. The argument goes as follows:

> STEP 1: *Give the quantity a name.*

We'll call it F for Fredericka:

$$F = 0.9999\ldots.$$

STEP 2: *Multiply by ten*

We obtain

$$10F = 9.9999\ldots.$$

STEP 3: *Subtract*

Now $10F - F$ is $9.9999\ldots - 0.9999\ldots$. Since the digits in the decimal places match this reads

$$9F = 9$$

giving

$$F = 1.$$

That is, the mathematics establishes that

$$0.9999\ldots = 1.$$

Lovely!

But let me be clear on what we have established here:

> ***IF*** *you choose to believe that* $0.9999\ldots$ *is a meaningful quantity in* ***USUAL MATHEMATICS****, then you must conclude that it equals* 1.

I say this because this algebraic argument can lead to philosophical woes.

Part 2: AN UNUSUAL NUMBER OUR USUAL MATHEMATICS MIGHT REJECT

The number $0.9999\ldots$ (if you choose to believe it is a one) has infinitely many 9s to the right of the decimal point. What if we consider the "number" with infinitely many 9s to the left of the decimal point instead?

$\ldots 9999$

This is a number that ends with nine. Actually it ends with ninety-nine. Actually it ends with nine-hundred-and-ninety-nine. And so on.

Let's apply our algebraic argument to see what value it must have.

STEP 1: *Give the quantity a name.*

We'll call it A for Allistaire:

$$A = ...9999\,.$$

STEP 2: *Multiply by ten*

We obtain

$$10A = ...99990\,.$$

STEP 3: *Subtract*

We see that A and $10A$ differ by nine (it is only their final digits that differ). Looking at $A - 10A$ we get

$$-9A = 9$$

giving

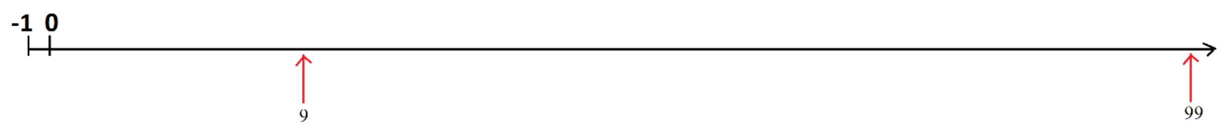
$$A = -1.$$

That is, our mathematics establishes that

$$...9999 = -1.$$

Apparently, if we pulled out a calculator and computed the sum $9 + 90 + 900 + 9000 + \cdots$ the calculator will show at the end of time the answer -1! Do you believe that?

Putting it another way: On a number line, do you believe that the numbers 9, 99, 999, 9999, ... are marching closer and closer to the number -1?



Challenge: Let's make matters worse! Consider the "number" with infinitely many 9s both to the left and to the right of the decimal point: ...9999.9999..... Use the same algebraic argument to show that this equals zero. (And this makes sense, because $...9999.9999.... = ...9999 + .9999... = -1 \ + 1 = 0.$)

It is hard to believe that $...9999$ is a meaningful number and, moreover, it has the value -1, at least in our usual way of think about arithmetic. After all, all we proved is: IF we choose to say that $...9999$ is meaningful, then it has value -1. It is up to us to decide whether or not it is meaningful quantity in the first place. Most people say it is not and stop there and that is fine.

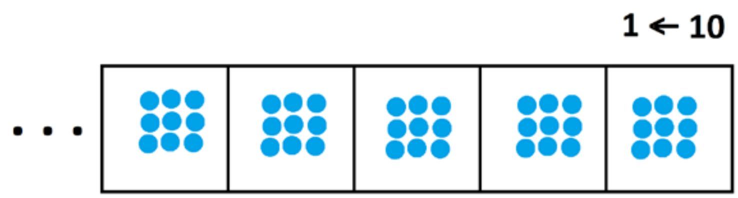
But this begs the question:

Is there an UNUSUAL system of arithmetic for which $...9999$ is meaningful (for which it has value -1)?

Challenge: One might be able to argue that $...9999$ does behave like -1 in ordinary arithmetic to some degree. For example, consider performing the (very) long addition shown. Do you see the answer zero results?

$$\begin{array}{lr} & ...999999 \\ + & 1 \\ \hline = & \end{array}$$

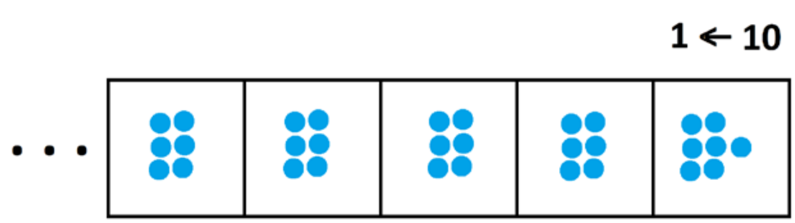
If you prefer, imagine what happens if you add one dot to this loaded $1 \leftarrow 10$ machine.



Challenge: Try this (very) long multiplication problem. Do you see that $...666667$ is behaving like the fraction $\frac{1}{3}$?

$$\begin{array}{lr} & ....66667 \\ \times & 3 \\ \hline = & \end{array}$$

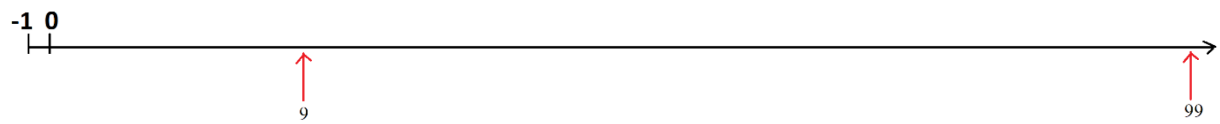
If you prefer, imagine what happens if you triple the count of dots in each of these boxes.



Extra: What “number” behaves like $\frac{2}{3}$?

Part 3: SOME UNUSUAL MATHEMATICS FOR UNUSUAL NUMBERS

It is possible to develop a arithmetic systems for numbers for which a number like $...9999$ is actually meaningful (and so has value -1 by our algebraic argument). Here's one approach that involves changing our sense of distance between numbers on the number line. It is one that will allow us to say, for instance, that the numbers 9, 99, 999, … are indeed marching closer and closer to -1 on the number line despite what our geometric training says!



Normal Distance on the Number Line

We usually say the number 5, for instance, is a distance five from 0 of the number line because 5 is five unit lengths from the zero. (We usually use absolute value notation for this distance: $|5| = 5$.)

And 3.7 is a distance 3.7 from 0, $|3.7| = 3.7$, because three-and-seven-tenths of a unit fit between 0 and 3.7 on the number line. And so on.

This is a very additive way of thinking about distance: adding five 1s get you from 0 to 5, adding 3.7 1s get you from 0 to 3.7, and so on. We can say that the distance of a point a on the number line, in this thinking, is the number of 1s that go additively into a.

But much of mathematics is not only concerned with the additive properties of numbers, but also the multiplicative properties of numbers. For example, we're interested in the prime factorizations of numbers (for example, $1000 = 2^3 5^3$ and $110 = 3 \cdot 5 \cdot 7$) and questions about prime factorizations in general are hard! (There are so many unanswered questions about the prime numbers in mathematics.)

Is there are way to bring the geometry of the number line into play here for these questions? Is there a way to think about the number line as perhaps structured multiplicatively rather than additively?

Rather than focus on all possible factors of numbers, let's focus on one possible factors of numbers. And to keep matters relevant to our base-ten arithmetic thinking, let's focus on the number 10.

In our additive thinking for distance on the number line we use the unit of 1 and ask how many ones (additively) go into each number for its distance from 0. We now want to use the unit of 10 and ask how many tens multiplicatively goes into each number.

What could that mean?

In the world of integers the number 0 is the most divisible number of all: it can be divided by any integer any number of times and still give an integer result (namely 0) each and every time. Focusing on our chosen factor of ten, we can divide 0 by ten once, or twice, or thirty-seven times, and still have an integer.

The number 40 is a little bit "zero-like" in this sense in that we can divide it by ten once and still have an integer. The number 1700 is more zero-like as it can be divided by ten twice and still give an integer result. A googol is very much more zero-like: it can be divided by ten one hundred times and still stay an integer.

The integer 5 is not very zero-like at all: one can't divide it by ten even once and stay an integer.

In this setting the more times ten "goes into" into a number multiplicatively, the more zero-like it is. So in this sense, a googol is much closer to zero than 5 is.

So let's develop a distance formula that regards numbers with large powers of ten as factors as closer to zero than numbers with less counts of powers of ten as factors. The following formula seems a natural attempt to do this.

If $a = 10^k m$ with m not divisibly by ten, then set $|a|_{ten} = \dfrac{1}{10^k}$.

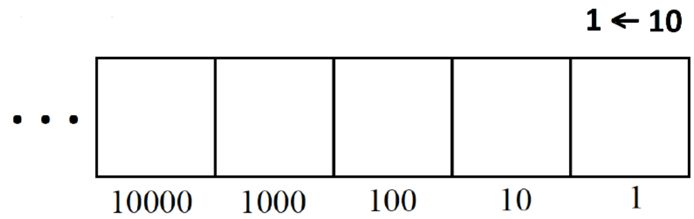
Since $300 = 10 \times 10 \times 3$ we have $|300|_{ten} = \dfrac{1}{100}$, and since $googol = 10^{100} \times 1$ we have $|googol|_{ten} = \dfrac{1}{10^{1000}}$. Also, $|5|_{ten} = \dfrac{1}{10^0} = 1$.

In particular, with this new way of measure distance, we see that

$$1, 10, 100, 1000, 10000, \ldots$$

is a sequence of numbers getting closer and closer to zero. (We have $|1|_{ten} = 1$ and $|10|_{ten} = 0.1$ and $|100|_{ten} = 0.01$ and $|1000|_{ten} = 0.001$, and so on, indeed approaching a distance of zero from 0.)

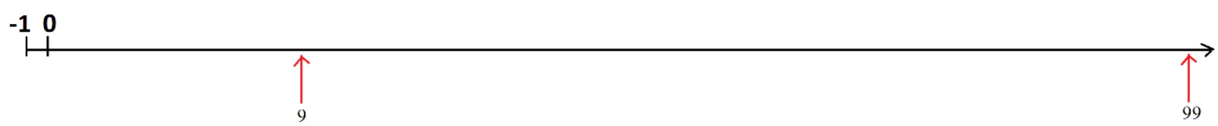
In terms of values in a $1 \leftarrow 10$ machine, we see that boxes far to the left in the machine, representing high powers of ten, are representing values very close to zero. (Before, boxes to the far right giving decimals were representing values very close to zero.)



Mathematicians call this way of viewing distances between the non-negative integers *ten-adic arithmetic*. (The suffix *adic* means "a counting of operations" and here we are counting factors of ten.) It is fun to think how to extend this notion of distance to fractions too, and then to all real numbers.

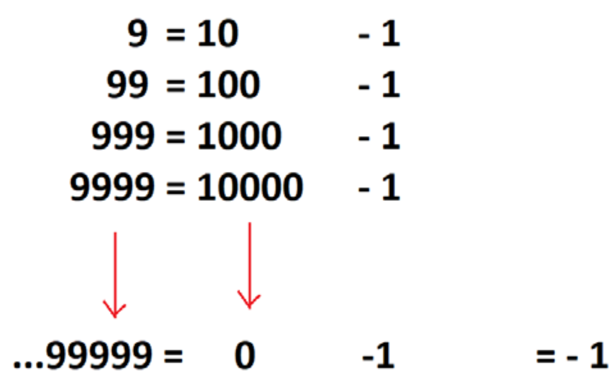
The number $\cdots 9999$

Let's look now at the sequence of numbers 9 and 99 and 999 and so on marching off to the right on the number line. Could they possibly be marching closer and closer to the value -1?



Yes, if by "closer" we mean this new multiplicative way to think of distance.

We have



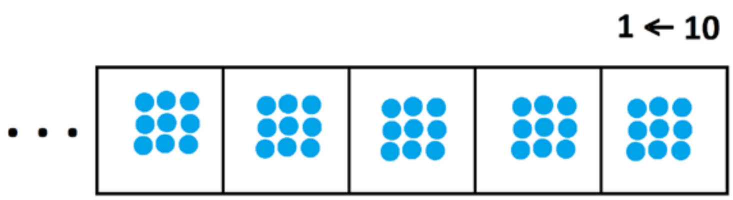
The numbers 9, 99, 999, 9999, … are indeed approaching the value $0-1=-1$.

Comment: The (very) long addition computation given below is justified in this way.

$$\begin{array}{r} \ldots 999999 \\ +\qquad\quad 1 \\ \hline = \qquad\quad\; \end{array}$$

We first compute $9+1=10$, and then we add 90 to this to obtain 100, and then we add 900 to obtain 1000, and so on. The further along we go with the computation the closer our results are to the number zero.

You can intuitively see this in the $1 \leftarrow 10$ machine: when you add one more dot to this loaded machine and perform the explosions, one clears away dots, pushing what remains further and further to the left where boxes have less and less significant value.

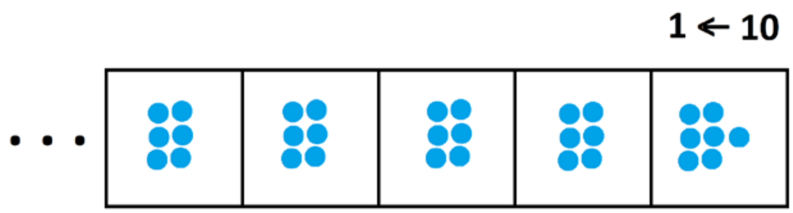


Computation of this next (very) long multiplication is justified in a similar way.

$$\begin{array}{r} \ldots.66667 \\ \times\qquad\quad 3 \\ \hline = \qquad\quad\; \end{array}$$

We first compute $3 \times 7 = 21$, and then add to this 3×60 to get 201, and then add to this 3×600 to get 2001, and so on. Now the numbers 20, 200, 2000, … are getting closer and closer to zero, so the numbers $21 = 20+1$, $201 = 200+1$, $2001 = 2000+1$, … are getting closer and closer to 1. The further along we go with this computation, the closer our answers are to the number 1.

Again, we can intuitively see this reasoning at play by tripling all the values in this loaded $1 \leftarrow 10$ machine.



Did you discover that $\cdots|3|3|3|3|4$ behaves like the fraction $\frac{1}{3}$?

Question: Doubling $\cdots|3|3|3|3|4$ gives $\cdots|6|6|6|6|8$ which is one more than $\cdots|6|6|6|6|7$, which is $\frac{1}{3}$. Is this consistent?

Challenge: Show that in a $2 \leftarrow 3$ machine that $\cdots 1|1|1|1|1|2$ is negative one! Show that $\cdots 0|1|0|1|0|1|0|2$ when multiplied by 5 gives 1, and so represents $\frac{1}{5}$. (What measure of distance might we be using on the number line this time for these "numbers" to make sense?)

Constructing Negative Integers

So far, in our base-ten thinking with our multiplicative notion of distance on the number line,

$$|10^k \times m| = \frac{1}{10^k} \text{ if } m \text{ has no factors of ten,}$$

we have made sense of $\cdots 9999$: it is a meaningful number and it has value -1.

So what's -2 in this unusual system of arithmetic?

Let's think in terms of a $1 \leftarrow 10$ machine. Since $-1 = \cdots 9|9|9|9$, and -2 is double -1, we should have

$$-2 = \cdots 18|18|18|18 .$$

With explosions we get

$$\begin{aligned} -2 &= \cdots 18|18|18|18 \\ &= \cdots 18|18|19|8 \\ &= \cdots 18|19|9|8 \\ &= \cdots 19|9|9|8 \\ &= \cdots 9|9|9|8 . \end{aligned}$$

And one can check that this long addition does give zero.

$$\begin{array}{lr} & \ldots 99998 \\ + & 2 \\ \hline = & \end{array}$$

We could also argue that

$$\begin{aligned} \cdots 99998 &= \cdots 99990 + 8 \\ &= \cdots 9999 \times 10 + 8 \\ &= (-1) \times 10 + 8 \\ &= -2 \end{aligned}$$

We now have some power! We can see that adding 47 to $\cdots 9999953$ will give zero and so this quantity must be -47, and that adding 3000 to $\cdots 99996000$ gives zero and so this quantity must be -3000.

Challenge: What is -2 in a $2 \leftarrow 3$ machine? What is -5?

Constructing Fractions

We saw that $\cdots 66667$ is the fraction $\frac{1}{3}$: multiply this quantity by three and you get 1.

The $1 \leftarrow 10$ machine provides a natural way to compute such fractions. For example, let's find the ten-adic representation of $\frac{4}{7}$. That is, let's find a number x such that $7 \times x = 4$. Start by writing

$$x = \cdots | h | g | f | e | d | c | b | a$$

as for a $1 \leftarrow 10$ machine. Then

$$7x = \cdots | 7h | 7g | 7f | 7e | 7d | 7c | 7b | 7a$$

We want $7a$, after explosions to leave a 4. So we need a multiple of 7 four greater than a multiple of 10. We see that $7a = 14$ is good. That is, we see we need $a = 2$.

$$x = \cdots | h | g | f | e | d | c | b | 2$$

$$\begin{aligned} 7x &= \cdots | 7h | 7g | 7f | 7e | 7d | 7c | 7b | 14 \\ &= \cdots | 7h | 7g | 7f | 7e | 7d | 7c | 7b + 1 | 4 \end{aligned}$$

Now we want $7b+1$ to be a multiple of 10 so that all dots in that box explode to leave zero behind. This suggests $b=7$.

$$x=\cdots|h|g|f|e|d|c|7|2$$

$$7x=\cdots|7h|7g|7f|7e|7d|7c|50|4$$
$$=\cdots|7h|7g|7f|7e|7d|7c+5|0|4$$

Now we need $7c+5$ a multiple of 10. Choose $c=5$.

$$x=\cdots|h|g|f|e|d|5|7|2$$

$$7x=\cdots|7h|7g|7f|7e|7d|40|0|4$$
$$=\cdots|7h|7g|7f|7e|7d+4|0|0|4$$

Now choose $d=8$.

$$x=\cdots|h|g|f|e|8|5|7|2$$

$$7x=\cdots|7h|7g|7f|7e|60|0|0|4$$
$$=\cdots|7h|7g|7f|7e+6|0|0|0|4$$

And then $e=2$.

$$x=\cdots|h|g|f|2|8|5|7|2$$

$$7x=\cdots|7h|7g|7f|20|0|0|0|4$$
$$=\cdots|7h|7g|7f+2|0|0|0|0|4$$

And $f=4$.

$$x=\cdots|h|g|4|2|8|5|7|2$$

$$7x=\cdots|7h|7g|30|0|0|0|0|4$$
$$=\cdots|7h|7g+3|0|0|0|0|0|4$$

And $g=1$.

$$x=\cdots|h|1|4|2|8|5|7|2$$

$$7x=\cdots|7h|10|0|0|0|0|0|4$$
$$=\cdots|7h+1|0|0|0|0|0|0|4$$

And now I am doing the same work as I did for a value b, making $7b+1$ is a multiple of 10. We are in a cycle and so

$$x=\frac{4}{7}=\cdots 142857\ 142857\ 142857\ 2=\overline{142857}\ 2.$$

Challenge: This process felt reminiscent of the task of writing $\frac{4}{7}$ as a decimal in ordinary arithmetic using a $1 \leftarrow 10$ machine with decimals. We argued there too that the decimal represent had to fall into a cycle.

Can you argue that the fraction $\frac{2}{13}$ too with having to have a repeating ten-adic expansion?

Challenge: What is the ten-adic expansion of $-\frac{4}{7}$?

One approach:

Write $-\frac{4}{7} = \overline{-1|-4|-2|-8|-5|-7}|-2$ and add some dots and antidot pairs to make all the terms positive.

$$\begin{aligned} -\frac{4}{7} &= \overline{-1|-4|-2|-8|-5|-7}|-2+(-8+8) \\ &= \overline{-1|-4|-2|-8|-5|-7-1}|8 \\ &= \cdots \end{aligned}$$

A glitch

Let's try to compute the ten-adic representation of the fraction $\frac{1}{2}$. Here we seek a number

$$x = \cdots|h|g|f|e|d|c|b|a$$

so that

$$2x = \cdots|2h|2g|2f|2e|2d|2c|2b|2a$$

equals 1.

This means we a number a so that, after explosions, $2a$ leaves a single dot. That is, we need $2a$ to be one more than a multiple of ten. This is not possible!

Challenge: Contemplate the ten-adic expansions for $\frac{1}{5}$ and $\frac{3}{10}$ and $\frac{2}{35}$.

In general, which fractions $\frac{p}{q}$ seem to be problematic?

Challenge: Develop a general theory that if $\frac{p}{q}$ is a reduced fraction with q sharing no factor in common with ten (other than 1), then it is for certain possible to express $\frac{p}{q}$ as a ten-adic number $\cdots hgfedcba$. Show further that its expression is sure to fall into a repeating cycle.

Broadening our definition a tad

It seems we have defined a ten-adic value to be an expression of the form $\cdots edcba$ with each digit one of the standard digits 0 through 9, allowing for non-zero digits to appear infinitely far to the left.

In this system we have the ordinary positive integers,

eg $5 = \cdots 00005$

the negative numbers

eg $-5 = \cdots 99995$

and some fractions

eg $\frac{1}{3} = \cdots 66667$.

But not all fractions. It turns out that the troublesome fractions are the ones $\frac{p}{q}$ which, when written in reduced form, have a denominator a multiple of 2 or 5 or both.

We can obviate this problem if we allow a ten-adic number to extend finitely far into the decimal places on the right. That is, set a ten-adic expression to be one of the form $\cdots edcba.xy\ldots z$ with each digit one of the standard digits 0 through 9, allowing for non-zero digits to appear infinitely far to the left of the decimal point, and only finitely far to its right.

Now we have

$$\frac{1}{2} = 0.5 = \cdots 00000.5$$

and

$$\frac{23}{100} = 0.23 = \cdots 000.23.$$

We can also handle $\frac{2}{35}$ by thinking of this as

$$\frac{2}{7\times5}=\frac{2\times2}{7\times10}=\frac{4/7}{10}.$$

Since $\frac{4}{7}=\overline{142857}\ 2$ we must have $\frac{2}{35}=\overline{142857}.2$.

Challenge: Show that $\frac{1}{6}=\cdots33333.5$ and hence find the ten-adic expression for $\frac{5}{12}$.

What is the ten-adic expression for $\frac{1}{12}$?

Challenge: Explain why every fraction is now sure to have a ten-adic representation.

Challenge: In ordinary arithmetic, the quantity $0.abcabcabc\cdots=0.\overline{abc}$ is the fraction $\frac{abc}{999}$. We see this by setting $x=0.abcabcabc\cdots$ and noticing that $1000x=abc.abcabcabc\cdots$. Subtracting then yields $999x=abc$.

Show that the same algebra applied to the ten-adic number $\cdots abc\,abc\,abc=\overline{abc}$ shows that it this number has value $-\frac{abc}{999}$.

Challenge: Explore a theory of "3/2-adic" representations of fractions using a $2\leftarrow3$ machine.