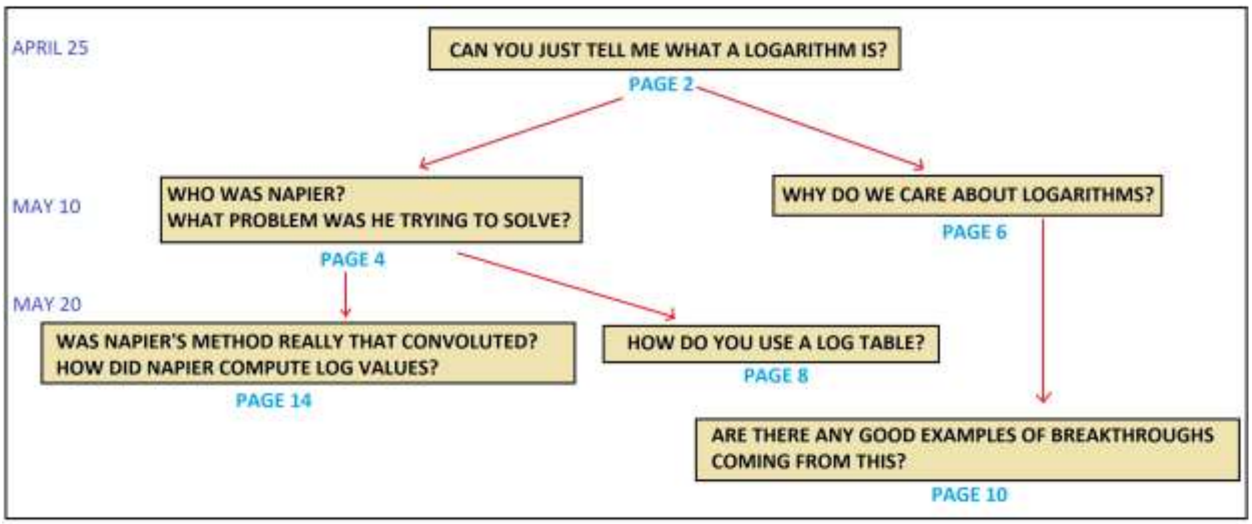
May 20

LOGARITHMS

Our knowledge map so far:



INSTRUCTIONS FOR MAY 30:

- Read the passages that are of interest to you in response to the new questions in this map. Reflect on them and choose two questions to send my may (at tanton.math@gmail.com.)

  Feel free to repeat a question you may have asked before that has not yet been answered, or ask a question someone else asked, or provide new questions.

- On May 30 I'll collate the questions and choose two or three that seem most pressing and respond to them. Our knowledge map of the mathematics, history, and teaching of logarithms is growing!

CAN YOU JUST TELL ME WHAT A LOGARITHM IS?

Response:

WHAT IS A LOGARITHM

Let's not worry about context for the moment or reasons why anyone in their right mind would want to study these things, and just get the very first issue out of the way right away. Let's just figure out what a logarithm is. We can worry about all that other stuff later on.

The mathematics of logarithms is actually, surprisingly, remarkably straightforward. Let's play a game.

Suppose I wrote on a board

$$power_2(8) = 3$$

and

$$power_5(25) = 2.$$

Do you think you could guess what is going on? (I am assuming we know about powers of numbers.)

Can you figure out each of these next examples?

$power_3(27) =$ ______

$power_{10}(100) =$ ______

$power_4(16) =$ ______

$power_4(64) =$ ______

$power_7\left(\frac{1}{7}\right) =$ ______

$power_2\left(\sqrt{2}\right) =$ ______

$power_{\frac{1}{3}}(9) =$ ______

$power_{10}(million) =$ ______

$power_{73}(1) =$ ______

$power_{0.01}(1000) =$ ______

$power_{100}(0.1) =$ ______

$power_{\sqrt{6}}\left(\frac{1}{36}\right) =$ ______

$power_1(5) =$ ______

The answers column-wise are: 3, 2, 2, 3 , -1, 1/2, -2 and then 6, 0, -3/2, -1/2, -4, and impossible! (The last few in the second column are tricky!)

Okay. We're done. We've just done logarithms!

Logarithms are just powers. But for very quirky historical reasons people don't use the word "power" as they should, but instead use the really scary made up word *logarithm*, shortened to just *log*. (There was this fellow by the name of Napier who invented these things. He was saving all of global science from a very annoying basic problem and did a great thing for the world by inventing these things. But no one realized at the time that what he was doing were just powers!)

$$\overset{\log}{\cancel{power}}_3(27) = \underline{3}$$
$$\overset{\log}{\cancel{power}}_{10}(100) = \underline{2}$$
$$\overset{\log}{\cancel{power}}_4(16) = \underline{2}$$
$$\overset{\log}{\cancel{power}}_4(64) = \underline{3}$$
$$\overset{\log}{\cancel{power}}_7\left(\frac{1}{7}\right) = \underline{-1}$$
$$\overset{\log}{\cancel{power}}_2\left(\sqrt{2}\right) = \underline{1/2}$$
$$\overset{\log}{\cancel{power}}_{\frac{1}{3}}(9) = \underline{-2}$$
$$\overset{\log}{\cancel{power}}_{10}(million) = \underline{6}$$
$$\overset{\log}{\cancel{power}}_{73}(1) = \underline{0}$$
$$\overset{\log}{\cancel{power}}_{0.01}(1000) = \underline{-3/2}$$
$$\overset{\log}{\cancel{power}}_{100}(0.1) = \underline{-1/2}$$
$$\overset{\log}{\cancel{power}}_{\sqrt{6}}\left(\frac{1}{36}\right) = \underline{-4}$$
$$\overset{\log}{\cancel{power}}_1(5) = \underline{\text{impossible}}$$

The point here is that **whenever you see the word** ***log*** **just think the word** *power***:**

$$\log_b(x)$$

is simply

$$power_b(x),$$

the power of b that gives the answer x.

WHO WAS NAPIER?
WHAT PROBLEM WAS HE TRYING TO SOLVE?

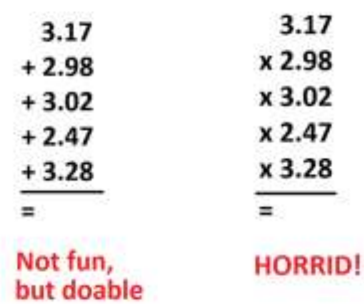
Response:

A BRIEF HISTORY OF LOGARITHMS

This material also appears in *https://www.youtube.com/watch?v=Cc5LhCyd8lM*.

During the Renaissance, science flourished in Europe. Scholars were collecting data and working with data to understand the world around them. And of course, they had to repeatedly perform arithmetic computations on large lists of data numbers to perform statistics, to analyze equations, and so on. But, of course, all this arithmetic had to be done with pencil and paper.

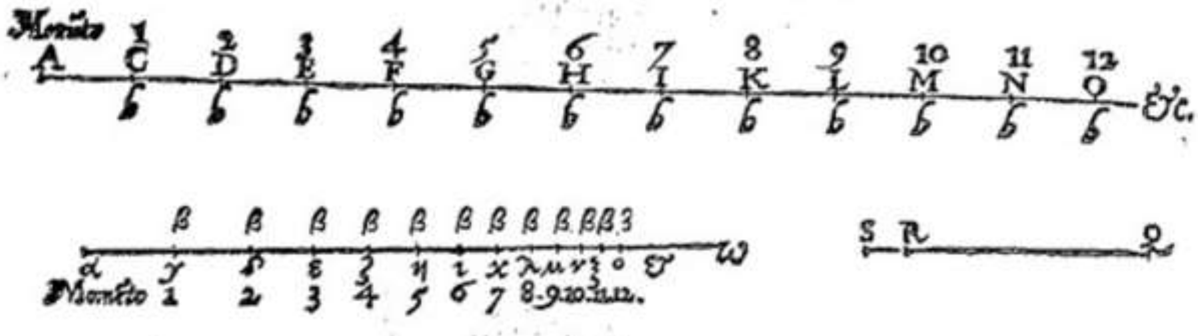
Now adding a large list of numbers is not fun, but it is doable. Multiplying a large list of numbers, on the other hand, is downright horrid.



The fact that multiplying numbers is so hard and so tedious actually held back scientific progress all through the 1400s and 1500s!

So a Scottish mathematician by the name of John Napier (1550 – 1617) set out to ease the tremendous woe of all science and invent a method that would turn multiplication problems into addition problems.

Napier was an inventive and creative fellow. After much toying and playing, he came up with mighty complex method that did the trick. To multiply to numbers, say M and N, Napier imagined two particles each moving along a number line, one an infinite line and one a finite line. The first particle moved at a uniform speed that was related to the number M, and the second at a speed, related to the number N and varying according to the distance it still had to traverse across the finite line.



He found that computing the ratio of the velocities of the two particles was a procedure that essentially turned the computation of $M \times N$ into an addition problem. It was complicated, and strange, but it worked! Napier also inserted a factor of 10,000,000 into all his computations to help out all the geoscientists that often had to work with large figures.

Napier invented a name for his method based on the Greek word *logos* for ratio and *arithmos* for number, hence logarithm.

No one really understood his method. So he, with the help of a colleague, Henry Briggs (1561 – 1630) decided to create tables of values – log tables – that scientists could simply refer to, without knowing the details behind his method, to convert products into sums with ease.

value	logarithm
1	0
2	.301
3	.477
4	.602
5	.699
6	.778
7	.845
8	.903
9	.954
10	1

To compute 2x3

log(2) = .301
log(3) = .477

Add .778

We see this matches the answer 6.

Napier's logarithms literally saved the progress of science.

It wasn't until another 200 years or so before scholar realized that Napier's logarithms were essentially "powers" backwards. But by then – and now another three hundred years later – the name *logarithm* had stuck.

More detail can be found here: http://www.maa.org/press/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-john-napier-introduces-logarithms

WHAT ARE YOUR NATURAL NEXT QUESTIONS IN RESPONSE TO THIS PASSAGE?

WHY DO WE CARE ABOUT LOGARITHMS?

Response:

Key Properties of Logarithms

Back in the 1600s, folk were so excited about logarithms because they unlocked a really simple arithmetic issue that was just holding back everything in science. John Napier, the inventor of logarithms, saved science back then!

Napier managed to find a way to convert multiplication problems -- really hard to do with pencil and paper -- into addition problems -- still not fun, but considerably simpler to do.

Napier managed to show that his logarithms satisfy the rule

$$\log_b\left(M\times N\right)=\log_b\left(M\right)+\log_b\left(N\right).$$

This property of logarithms was a savior in the 1600s, 1700s, and 1800s. But today, with calculators, doing multiplication computations is not an issue!

So why do we care about logarithms today?

If you take Napier's multiplication rule and look at it with $M=N$, you get

$$\log_b\left(M^2\right)=\log_b\left(M\right)+\log_b\left(M\right)=2\log_b\left(M\right).$$

And again with the numbers M and M^2,

$$\log_b\left(M^3\right)=\log_b(M\times M^2)=\log_b\left(M\right)+2\log_b\left(M\right)=3\log_b\left(M\right).$$

And so on.

This suggests the property

$$\log_b\left(M^x\right)=x\log_b\left(M\right).$$

If this rule really is true, it suggests a way to bring a variable in an equation that is locked in as an exponent down to a level that is manageable.

Example: *Assuming this property of logarithms is true, solve* $7^x = 5^{x+2}$.

With all the tools and techniques one learns in algebra class before a study of logarithms, a question like this is impossible to solve! (Variables stuck upstairs are just stuck.)

But let's "hit" both sides of the equation with a log. We can shake those exponents down.

From

$$\log_b\left(7^x\right) = \log_b\left(5^{x+2}\right)$$

we get

$$x\log_b(7) = (x+2)\log_b(5)$$

and so

$$x\log_b(7) = x\log_b(5) + 2\log_b(5)$$

giving

$$x = \frac{2\log_b(5)}{\log_b(7) - \log_b(5)}.$$

This looks horrible! Plus we haven't specified which b to use. (Apparently this answer is the same no matter which value of b we choose. Whoa!)

If you look at a calculator, you see a "log" button without any mention of base. Henry Briggs was a fellow who helped out Napier by suggesting that since we work in base 10 in arithmetic, perhaps we should work with base 10 in logarithms too. He did all the computations of log-base-10 values for Napier, and today logarithms with this base are called *Briggsian Logarithms* in his honor. When a base is not mentioned for a logarithm, as for calculators, it is assumed that the logarithm is Briggsian, that is, base 10.

So we have

$$x = \frac{2\log(5)}{\log(7) - \log(5)}$$

which, using a calculator, gives

$$x \approx \frac{2\times 0.699}{0.845 - 0.699} \approx 1.252.$$

This is a contrived example, but often in modern mathematics one is trying to solve an equation with the variable stuck upstairs. The way to handle this is to **hit the equation with a log** (on both sides) **and shake the variables down.**

WHAT ARE YOUR NATURAL NEXT QUESTIONS IN RESPONSE TO THIS PASSAGE?

HOW DO YOU USE A LOG TABLE?

Understanding Log tables

Scottish mathematician John Napier (1550 - 1617) wanted to help the scientific world by providing a simple means to do the arithmetic of multiplication, which is very hard to do by hand and is prone to errors. (Which seems more fun to do: computing 3.156 + 6.279 or computing 3.156 x 6.279?)

He found a method for converting multiplication problems into addition problems, but it was difficult to understand. His colleague Henry Briggs (1561 - 1630) suggested a simplification of his approach, which then led to the production of "log tables" for scientists to use, ones more convenient than the ones Napier first provided.

Here's what one looks like.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1·0	·0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
1·1	·0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
1·2	·0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
1·3	·1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
1·4	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1·5	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
1·6	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
1·7	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1·8	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
1·9	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
2·0	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2·1	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2·2	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
2·3	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
2·4	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
2·5	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
2·6	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15

Each value in the table is a base-ten logarithm value. For example, we see in the table that $\log_{10}(1.40) = 0.1461$, $\log_{10}(1.41) = 0.1492$, and so on. Each answer is rounded to four decimal places.

This table actually offers you a bit more precision. To see the value of $\log_{10}(1.415)$, for instance, look at the number on the 1.4 row and under the digit 5 of the right section to see "15." This means we need to append the digits 1 and 5 to the final two digits of our answer to $\log(1.41)$.

$$\log_{10}(1.415) = \log_{10}(1.41) + 0.0015 = 0.1492 + 0.0015 = 0.1507.$$

But the point of these tables is that you don't need to know what any of these numbers actually mean! They simply let you compute multiplications.

For example, let's look up the answer to 1.41×1.22.

Now, according to the table 1.41 gives the number 0.1492 and 1.22 gives the number 0.0864. Add these.

$$0.1492+0.0864=0.23560.1492+0.0864=0.2356$$

Looking at back at the table we see that 0.2355 appears as the table value for 1.72. This is close to 0.2356, so our product we seek must be close to 1.72. (The correct answer is 1.41 x 1.22 = 1.7202!)

Actually, on the table, we see that $\log_{10}(1.72) = 0.2355$ and $\log_{10}(1.721) = 0.2357$ so our answer indeed 1.72 to two decimal places.

Scientists did not know what the numbers in the table meant, but they could use the tables to compute products of numbers quickly. This freed up science in the 1600s and allowed scholars to work with data and scientific measurements with relative ease. Napier, and Briggs, provided a profound service to furthering scientific scholarship.

Comment: The log values of numbers below 10 are all "zero point something;" the log value of all numbers between 10 and 100 are all "one point something;" between 100 and 1000 all "two point something;" and so on. Log tables will only show the decimal part of log values, assuming you, the reader, already know what the integer part should be.

WHAT ARE YOUR NATURAL NEXT QUESTIONS IN RESPONSE TO THIS PASSAGE?

One of mine is:

What is "ln" on my calculator?

ARE THERE ANY GOOD EXAMPLES OF BREAKTHROUGHS COMING FROM THIS?

A Variety of Uses of Logarithms

First of all, logarithms give a sense of scale of really big numbers.

A 1 followed by six zeros represents a number we call a *million*; a 1 followed by nine zeros represents a number we call a *billion*; a 1 followed by 12 zeros represents a number we call a *trillion*. We have some vague sense of meaning of these numbers because we often speak of money and national debt in terms of millions, billions, and trillions.

But keep adding to the number of zeros trailing a 1 and our intuition starts to fail us. A 1 followed by 15 zeros is one-thousand-fold as big as a trillion, we call that a *quadrillion*. And then keeping track of names of numbers start to feel awkward and silly: it becomes easier to just think in terms of the number of zeros we are dealing with, that is, in terms of the powers of ten at hand.

$10^6 =$ million

$10^9 =$ billion

$10^{12} =$ trillion

$10^{15} =$ quadrillion

$10^{80} =$ the approximate number of molecules in the universe.

That is, speaking in terms of powers (logarithms) starts to give us a means to work with and handle large numbers.

The number of digits of a number

In computer science, numbers are stored in memory using their base-two representations with only the digits 0 and 1. So, how many digits are needed to store the number 10^{80} in memory?

In base 10, the answer is easy: 10^{80} is $1+80=81$ digits long. Continuing with base-ten thinking for a moment, the four-digits 6382, for instance, represent the number

$$6\times10^3+3\times10^2+8\times10+2\times1.$$

In fact every four-digit number N is a number between 10^3 and 10^4.

$$10^3 \le N < 10^4$$

(The smallest N can be is $1000 = 10^3$ and the largest it can be is 9999 which is one below 10^4.)

"Hitting" this inequality with a log, and using the standard logarithm rules, gives

$$\log\left(10^3\right) \le \log(N) < \log\left(10^4\right),$$

that is, $3 \le \log(N) < 4$.

This shows that if we take the (base ten) logarithm of N and round it down to the nearest integer, we'd get one less than the number of digits the number has. In general

The number of digits N *has, in base ten, is* $\lfloor \log(N) \rfloor + 1$*.*

So, basically, $\log(N)$ is the number of digits N has, adjusted a bit to an integer value. (The $\lfloor\ \rfloor$ brackets mean "round down to the nearest integer.")

There is nothing special here about base ten arithmetic here.

The number of digits N *has in base* b *is* $\lfloor \log_b(N) \rfloor + 1$*.*

For example, the number 6382 has $\lfloor \log_5(6382) \rfloor + 1 = 5 + 1 = 6$ digits in base 5 and $\lfloor \log_3(6382) \rfloor + 1 = 7 + 1 = 8$ digits in base 3.

(As a check, $5^5 = 3125 < 6382 < 5^6 = 15625$ and so 6328 can be written as some combination of the powers 5^5, 5^4, ..., 5^1, and 1, and so is indeed six digits long in base five.)

And to answer our original question, the number 10^{80} is $\lfloor \log_2\left(10^{80}\right) \rfloor + 1 = 266$ digits long in base two and so requires 266 bits of memory to store.

Order of Magnitude

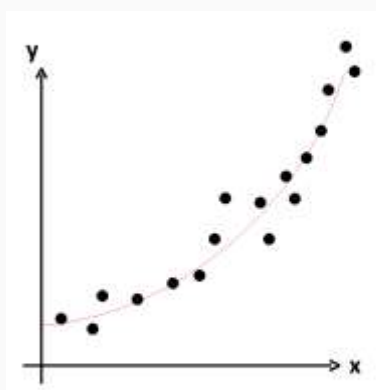
If a number M has value between 10^N and 10^{N+1}, then we say that it is "of the order of magnitude" 10^N. For example, a number in the millions has order of magnitude 10^6 and the number of molecules in a mole is of order of magnitude 10^{23}.

The forces of Earthquakes are large-scale numbers that are difficult to comprehend. We find it easier to think of the order of magnitude of Earthquakes rather than the exact values of the forces involved. The Richter scale is one that represents a scale of order of magnitudes: an Earthquake that has measure 6.2 on the Richter scale is ten times more forceful than one measuring 5.2 and a hundred times more forceful than one measuring 4.2 on the scale. We're naturally working with a logarithmic scale.

Handling Exponential Growth

Numbers in natural systems grow in value rapidly if they obey a principle of multiplicative growth, that is, if the size of a quantity or population one time period is a fixed multiple of its size the previous period. For example, if a bacteria population doubles in mass every hour, then 1 gram of mass becomes $2^{24} = 16,777,216$ grams of mass in just 24 hours.

Of course, in real world observation, the exact growth rate of a particular system is hard to pin down, even if data from an observation seems to follow exponential growth.



How does one find a "curve of best fit" for a system that seems to be following exponential growth? That is, how does one determine the appropriate values of a and b to give an equation of the form

$$y = ab^x$$

that fits a scatter plot well?

Scientists have long had the good technique for finding lines of best fit for scatter plots that seem to suggest a liner relationship – the method of least squares. Can that method be used here too?

Yes!

Apply a logarithm to the desired equation, making use of the standard log rules,

$$\log(y) = x\log(b) + \log(a)$$

and work with the data of x values and $\log(y)$ values instead. They follow a linear relationship. Find a best-fit linear values of $\log(b)$ and $\log(a)$, and then compute from them a and b. We now have an equation $y = ab^x$ that fits the exponential data.

WHAT ARE YOUR NATURAL NEXT QUESTIONS IN RESPONSE TO THIS PASSAGE?

One of mine is:

WHAT ARE THE STANDARD LOG RULES?

WAS NAPIER'S METHOD REALLY THAT CONVOLUTED? HOW DID NAPIER COMPUTE LOG VALUES?

How Napier Computed his Log Values

Scottish mathematician John Napier (1550 - 1617) toiled for many decades developing a technique that would provide a simple means for converting multiplication problems into addition problems. Today we understand his technique in terms of the powers of numbers: to multiply 10^a and 10^b, simply add the exponents. But this is not the approach Napier took.

Mathematical historians are not clear what lead Napier to work with a dynamic model—comparing the motions of two particles each on a straight line—and it took scholars of the time many decades to realise that Napier's logarithms could be defined in terms of exponents. John Wallis in 1685 and then Johann Bernoulli in 1694 were the first to start seeing connections along these lines, but the mathematics of (the equivalent of) fractional exponents was still not properly understood. It was not until the work of Swiss mathematician Leonhard Euler that a definitive theory of exponents was in hand and logarithms were finally seen for what they truly are. But by then, Napier's name *logarithm* for these exponents-in-disguise was entrenched.

Computing Logarithms

Before we delve into the mathematics of Napier's approach, let's see how it is possible to compute some logarithmic values by hand with a clever choice of base. (We'll use the modern notation of exponents here.)

Set $b = \left(1 - \frac{1}{10}\right)^{10}$, then

$$b^{0.1} = \left(1 - \frac{1}{10}\right)^{1} = 0.9$$

$$b^{0.2} = b^{0.1} \times b^{0.1} = 0.9 \times \left(1 - \frac{1}{10}\right) = 0.9 - 0.09 = 0.81$$

$$b^{0.3} = b^{0.2} \times b^{0.1} = 0.81 \times \left(1 - \frac{1}{10}\right) = 0.81 - 0.081 = 0.729$$

We see that each term is the previous value minus one tenth of the previous value. Thus $b^{0.1}, b^{0.2}, b^{0.3}, b^{0.4}, \ldots$ can all be readily computed by hand. We have

$$\log_b(0.9) = 0.1$$
$$\log_b(0.81) = 0.2$$
$$\log_b\left(0.729\right) = 0.3$$
$$\ldots$$

If we work with $b = \left(1 - \frac{1}{100}\right)^{100}$, we can readily work out $b^{0.01}$, $b^{0.02}$, $b^{0.03}$, … by hand, and thus some logarithm values to two-decimal places.

Napier, in his work, decided to work to seven decimal places and thus was working with the equivalent of a base of

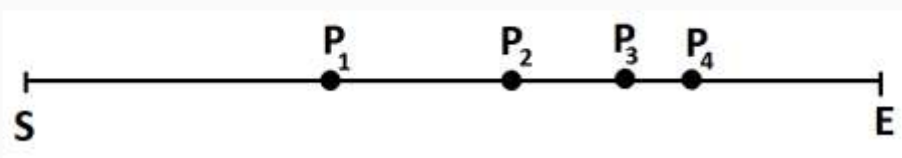
$$b = \left(1 - \frac{1}{10^7}\right)^{10^7}.$$

Towards Napier’s Approach

Here is the gist of Napier’s kinetic approach, developed after several decades of deep thinking, no doubt.

Napier envisioned a particle moving along a line segment, starting at one endpoint S and heading towards the other, E, and doing so in a way that its velocity decreased in a proportional way:

> *If, in equal time intervals, the particle moves from S to P_1 to P_2 and so on, then the proportions $\frac{SP_1}{SE} = \frac{P_1P_2}{P_1E} = \frac{P_2P_3}{P_2E} = \cdots$ are equal.*



So if the particle moves one-sixth of the way in the first time period, say, then it moves one-sixth of the distance that remains in the next time period, one-sixth of what then remains the next time period, and so on. The velocity of the particle thus decreases with time and it approaches a velocity of zero as it nears E.

Suppose this particle starts with velocity v.

Then Napier imagined a second particle moving along an infinitely long line at constant speed v. This particle thus moves the same distance over equal time intervals.

Napier then, essentially, defined his logarithm of a number x as follows:

Compute the time t it takes for the particle reach the point P that is x units from E. Here

$$PE = x.$$

Then the Napier Logarithm of x is the distance vt the particle moving at constant velocity moves in this time period.

In order to compute times and distances in such a scenario, Napier set some convenient values. He set the line segment to be

$$SE = 10^7$$

units long, and chose an initial velocity of the particle so that it moves $\frac{1}{10^7}$ of the way each second. Thus

$$\frac{SP_1}{SE} = \frac{P_1P_2}{P_1E} = \frac{P_2P_3}{P_2E} = \cdots = \frac{1}{10^7}.$$

He was able to compute velocities and distances traveled for each whole number of seconds. From $\frac{P_nP_{n+1}}{P_nE} = \frac{1}{10^7}$ we get

$$\frac{P_nE - P_{n+1}E}{P_nE} = \frac{1}{10^7}$$

giving

$$P_{n+1}E = \left(1 - \frac{1}{10^7}\right)P_nE.$$

Starting with $SE = 10^7$ we thus get

$$P_1E = \left(1 - \frac{1}{10^7}\right)SE = \left(1 - \frac{1}{10^7}\right)10^7$$

and each value P_nE is the previous value minus $1/10^7$ of that value. Thus, in N seconds, the particle is at a position that is $P_NE = \left(1 - \frac{1}{10^7}\right)^N \times 10^7$ units away from the endpoint E, So, according to Napier's definition, if $x = \left(1 - \frac{1}{10^7}\right)^N \times 10^7$, then

$$NapierLog(x) = N.$$

As the previous section showed, we can compute such values with ease. This thus gave Napier a whole "bank" of log values to work with which he could then use interpolation and other mathematical techniques to estimate intermediate logarithm values. (This was by no means easy work and the inaccuracies that arose from estimations caused problems for scientists attempting to do higher and higher precision calculations. Many scholars throughout the 1600s worked hard to create more and more accurate tables of log values.)

In modern notation, we see from $x = \left(1 - \frac{1}{10^7}\right)^N \times 10^7$ that

$$NapierLog(x) = \log_{1-\frac{1}{10^7}}\left(\frac{x}{10^7}\right).$$

Finishing Napier's Approach

At Napier's time astronomers were using extensive tables of sine and cosine values and many of their calculations involved multiplying and dividing these trigonometric values. To address the concerns of astronomers, in particular, Napier assumed that his number x, the distance of a point x from the endpoint along the segment SE, was actually a trigonometric value $\sin(\theta)$ for some angle θ, usually multiplied by a large power of ten to help avoid decimals,

$$x = 10^9 \sin(\theta).$$

So Napier actually gave tables of "logarithms of angles," which, in modern notation, would be equivalent to something like

$$NapierLog(\theta) = \log_{1-\frac{1}{10^7}}\left(\frac{10^9 \sin(\theta)}{10^7}\right).$$

To actually see that Napier's approach is equivalent to computing a logarithm as we know it, one must use the techniques of calculus. These techniques were not available to Napier at the time, but he developed intuitive arguments, which turned out to be correct, to justify some of the assumptions he made about these moving particles.

For instance, Napier argued that a particle that moved equal proportions of distances remaining along the line segment SE during equal time intervals must actually have the property that its velocity at any point P is proportional to its remaining distance from the endpoint E. In calculus this allows us to write and solve a differential equation which shows that the particle at any time t is at a distance Ae^{kt} units from endpoint E for some constants A and k. Computing the time taken to reach a particular position along the line, that is, given the value Ae^{kt} and computing t from it, is equivalent to computing a logarithm.

WHAT ARE YOUR NATURAL NEXT QUESTIONS IN RESPONSE TO THIS PASSAGE?

One of mine is:

Does the curriculum define logarithms as inverse functions or something?