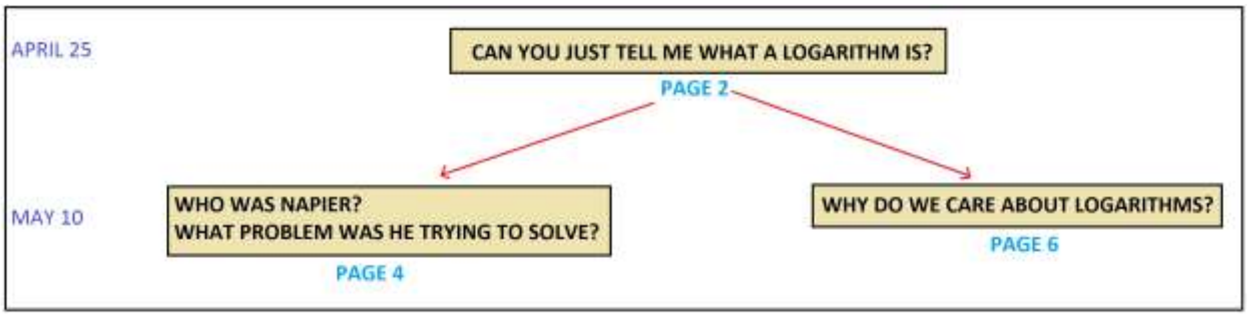
May 10

LOGARITHMS

Our knowledge map so far:



INSTRUCTIONS FOR MAY 20:

- Read the two passages in response to the next two questions in this map. Reflect on them and choose two questions to send my may (at tanton.math@gmail.com.)

  Feel free to repeat a question you may have asked before that has not yet been answered, or ask a question someone else asked or provide new questions.

- On May 20 I'll collate the questions and choose two or three that seem most pressing and respond to them. We'll repeat this process and work to create a knowledge map about the mathematics, history, and teaching of logarithms.

This map and this document will grow over time. It will represent our own organic, human journey towards fully understanding the topic and its nuances.

CAN YOU JUST TELL ME WHAT A LOGARITHM IS?

Response:

WHAT IS A LOGARITHM

Let's not worry about context for the moment or reasons why anyone in their right mind would want to study these things, and just get the very first issue out of the way right away. Let's just figure out what a logarithm is. We can worry about all that other stuff later on.

The mathematics of logarithms is actually, surprisingly, remarkably straightforward. Let's play a game.

Suppose I wrote on a board

$$power_2(8) = 3$$

and

$$power_5(25) = 2.$$

Do you think you could guess what is going on? (I am assuming we know about powers of numbers.)

Can you figure out each of these next examples?

$$power_3(27) = \_\_\_\_\_ \qquad power_{10}(million) = \_\_\_\_\_$$
$$power_{10}(100) = \_\_\_\_\_ \qquad power_{73}(1) = \_\_\_\_\_$$
$$power_4(16) = \_\_\_\_\_ \qquad power_{0.01}(1000) = \_\_\_\_\_$$
$$power_4(64) = \_\_\_\_\_ \qquad power_{100}(0.1) = \_\_\_\_\_$$
$$power_7\left(\frac{1}{7}\right) = \_\_\_\_\_ \qquad power_{\sqrt{6}}\left(\frac{1}{36}\right) = \_\_\_\_\_$$
$$power_2\left(\sqrt{2}\right) = \_\_\_\_\_ \qquad power_1(5) = \_\_\_\_\_$$
$$power_{\frac{1}{3}}(9) = \_\_\_\_\_$$

The answers column-wise are: 3, 2, 2, 3 , -1, 1/2, -2 and then 6, 0, -3/2, -1/2, -4, and impossible! (The last few in the second column are tricky!)

Okay. We're done. We’ve just done logarithms!

Logarithms are just powers. But for very quirky historical reasons people don't use the word "power" as they should, but instead use the really scary made up word *logarithm*, shortened to just *log*. (There was this fellow by the name of Napier who invented these things. He was saving all of global science from a very annoying basic problem and did a great thing for the world by inventing these things. But no one realized at the time that what he was doing were just powers!)

log ~~*power*~~$_3(27) =$ **3**

log ~~*power*~~$_{10}(100) =$ **2**

log ~~*power*~~$_4(16) =$ **2**

log ~~*power*~~$_4(64) =$ **3**

log ~~*power*~~$_7\left(\frac{1}{7}\right) =$ **-1**

log ~~*power*~~$_2\left(\sqrt{2}\right) =$ **1/2**

log ~~*power*~~$_{\frac{1}{3}}(9) =$ **-2**

log ~~*power*~~$_{10}(million) =$ **6**

log ~~*power*~~$_{73}(1) =$ **0**

log ~~*power*~~$_{0.01}(1000) =$ **-3/2**

log ~~*power*~~$_{100}(0.1) =$ **-1/2**

log ~~*power*~~$_{\sqrt{6}}\left(\frac{1}{36}\right) =$ **-4**

log ~~*power*~~$_1(5) =$ **impossible**

The point here is that **whenever you see the word** *log* **just think the word** *power***:**

$$\log_b(x)$$

is simply

$$power_b(x),$$

the power of b that gives the answer x.

WHO WAS NAPIER?
WHAT PROBLEM WAS HE TRYING TO SOLVE?

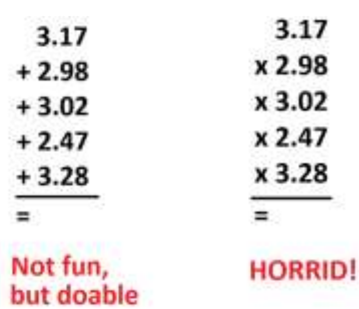
Response:

A BRIEF HISTORY OF LOGARITHMS

This material also appears in https://www.youtube.com/watch?v=Cc5LhCyd8lM.

During the Renaissance, science flourished in Europe. Scholars were collecting data and working with data to understand the world around them. And of course, they had to repeatedly perform arithmetic computations on large lists of data numbers to perform statistics, to analyze equations, and so on. But, of course, all this arithmetic had to be done with pencil and paper.

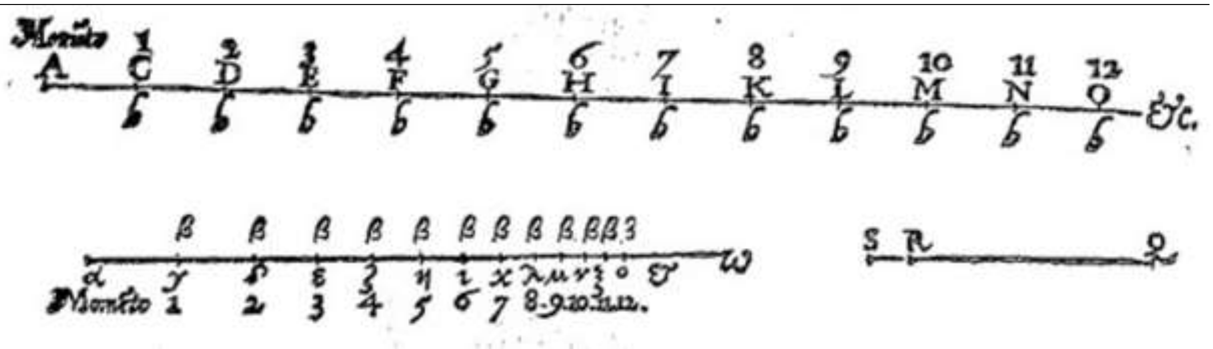
Now adding a large list of numbers is not fun, but it is doable. Multiplying a large list of numbers, on the other hand, is downright horrid.



The fact that multiplying numbers is so hard and so tedious actually held back scientific progress all through the 1400s and 1500s!

So a Scottish mathematician by the name of John Napier (1550 – 1617) set out to ease the tremendous woe of all science and invent a method that would turn multiplication problems into addition problems.

Napier was an inventive and creative fellow. After much toying and playing, he came up with mighty complex method that did the trick. To multiply to numbers, say M and N, Napier imagined two particles each moving along a number line, one an infinite line and one a finite line. The first particle moved at a uniform speed that was related to the number M, and the second at a speed, related to the number N and varying according to the distance it still had to traverse across the finite line.



He found that computing the ratio of the velocities of the two particles was a procedure that essentially turned the computation of $M \times N$ into an addition problem. It was complicated, and strange, but it worked! Napier also inserted a factor of 10,000,000 into all his computations to help out all the geoscientists that often had to work with large figures.

Napier invented a name for his method based on the Greek word *logos* for ratio and *arithmos* for number, hence logarithm.

No one really understood his method. So he, with the help of a colleague, Henry Briggs (1561 – 1630) decided to create tables of values – log tables – that scientists could simply refer to, without knowing the details behind his method, to convert products into sums with ease.

value	logarithm
1	0
2	.301
3	.477
4	.602
5	.699
6	.778
7	.845
8	.903
9	.954
10	1

To compute 2x3

log(2) = .301
log(3) = .477

Add .778

We see this matches the answer 6.

Napier's logarithms literally saved the progress of science.

It wasn't until another 200 years or so before scholar realized that Napier's logarithms were essentially "powers" backwards. But by then – and now another three hundred years later – the name *logarithm* had stuck.

More detail can be found here: http://www.maa.org/press/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-john-napier-introduces-logarithms

WHAT ARE YOUR NATURAL NEXT QUESTIONS IN RESPONSE TO THIS PASSAGE?

WHY DO WE CARE ABOUT LOGARITHMS?

Response:

Key Properties of Logarithms

Back in the 1600s, folk were so excited about logarithms because they unlocked a really simple arithmetic issue that was just holding back everything in science. John Napier, the inventor of logarithms, saved science back then!

Napier managed to find a way to convert multiplication problems -- really hard to do with pencil and paper -- into addition problems -- still not fun, but considerably simpler to do.

Napier managed to show that his logarithms satisfy the rule

$$\log_b\left(M \times N\right) = \log_b\left(M\right) + \log_b\left(N\right).$$

This property of logarithms was a savior in the 1600s, 1700s, and 1800s. But today, with calculators, doing multiplication computations is not an issue!

So why do we care about logarithms today?

If you take Napier's multiplication rule and look at it with $M = N$, you get

$$\log_b\left(M^2\right) = \log_b\left(M\right) + \log_b\left(M\right) = 2\log_b\left(M\right).$$

And again with the numbers M and M^2,

$$\log_b\left(M^3\right) = \log_b(M \times M^2) = \log_b\left(M\right) + 2\log_b\left(M\right) = 3\log_b\left(M\right).$$

And so on.

This suggests the property

$$\log_b\left(M^x\right) = x\log_b\left(M\right).$$

If this rule really is true, it suggests a way to bring a variable in an equation that is locked in as an exponent down to a level that is manageable.

Example: *Assuming this property of logarithms is true, solve* $7^x = 5^{x+2}$.

With all the tools and techniques one learns in algebra class before a study of logarithms, a question like this is impossible to solve! (Variables stuck upstairs are just stuck.)

But let's "hit" both sides of the equation with a log. We can shake those exponents down.

From

$$\log_b\left(7^x\right) = \log_b\left(5^{x+2}\right)$$

we get

$$x\log_b(7) = (x+2)\log_b(5)$$

and so

$$x\log_b(7) = x\log_b(5) + 2\log_b(5)$$

giving

$$x = \frac{2\log_b(5)}{\log_b(7) - \log_b(5)}.$$

This looks horrible! Plus we haven't specified which b to use. (Apparently this answer is the same no matter which value of b we choose. Whoa!)

If you look at a calculator, you see a "log" button without any mention of base. Henry Briggs was a fellow who helped out Napier by suggesting that since we work in base 10 in arithmetic, perhaps we should work with base 10 in logarithms too. He did all the computations of log-base-10 values for Napier, and today logarithms with this base are called *Briggsian Logarithms* in his honor. When a base is not mentioned for a logarithm, as for calculators, it is assumed that the logarithm is Briggsian, that is, base 10.

So we have

$$x = \frac{2\log(5)}{\log(7) - \log(5)}$$

which, using a calculator, gives

$$x \approx \frac{2 \times 0.699}{0.845 - 0.699} \approx 1.252.$$

This is a contrived example, but often in modern mathematics one is trying to solve an equation with the variable stuck upstairs. The way to handle this is to **hit the equation with a log** (on both sides) **and shake the variables down.**

WHAT ARE YOUR NATURAL NEXT QUESTIONS IN RESPONSE TO THIS PASSAGE?