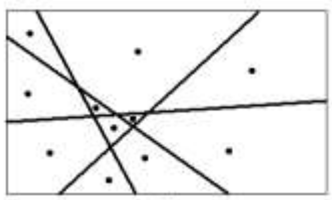
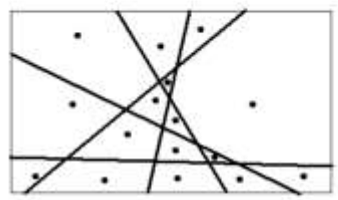
MORE WITHOUT WORDS

Mathematical Puzzles to Confound and Delight

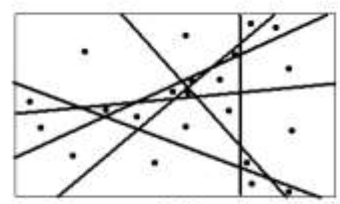
MWW 7: SOLUTION



11



16



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We are counting the maximum number of regions created by N lines crossing a page. (No three lines then can pass through a common point.) We have the sequence of numbers 1, 2, 4, 7, 11, 16, ….
It seems the numbers in this sequence differ by 1, 2, 3, 4, 5, … in turn and we have

$$1 = 1$$
$$2 = 1 + 1$$
$$4 = 1 + 1 + 2$$
$$7 = 1 + 1 + 2 + 3$$
$$11 = 1 + 1 + 2 + 3 + 4$$
$$16 = 1 + 1 + 2 + 3 + 4 + 5$$

and so on. We learned in WW15 the formula $1 + 2 + 3 + \cdots + N = \frac{N^2 + N}{2}$ suggesting the terms in the our sequence are given by $1 + \frac{N^2 + N}{2}$. But all this rests on knowing that if there are R regions in a picture with N lines, then there are $R + N + 1$ regions in a next picture with $N + 1$ lines.

To see why this is so, imagine drawing a new line in a picture of N lines, starting at one side of the page. When this new line first hits a previously drawn line, it splits a region in two and thereby increases the count of regions by 1. In fact this will be the case for each line the new line hits. To get the maximal number of regions we want the new line to hit all N previously drawn lines, thereby increasing the count of regions by N . The count increases one more time when the new line hits the other edge of the page as it splits one more region in two. We thus do indeed have $R + N + 1$ regions in a maximal picture of $N + 1$ lines.

Question: Is it always possible for a new line to intersect each and every previously drawn line?