11

INTEGER TRIANGLES

James Tanton

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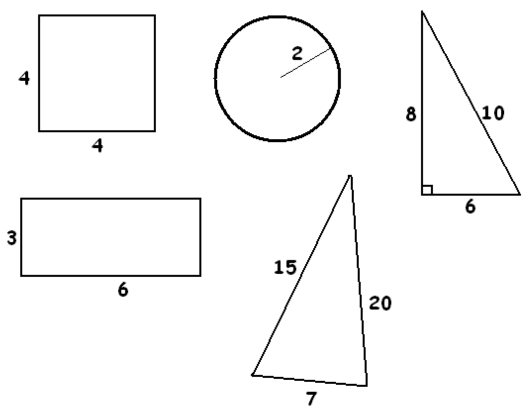
The St. Mark's
Institute of Mathematics
Newsletter

MAY

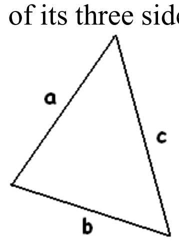
THIS MONTH'S PUZZLER: MORE THAN JUST AN AREA!

What property do each of the following figures share?



Find another right triangle (with integer sides) with the property. Is there another integer rectangle with the property?

HERON'S FORMULA: In 100 C.E. Heron of Alexandria (also known as Hero of Alexandria) published a remarkable formula for the area of a triangle in terms of its three side-lengths



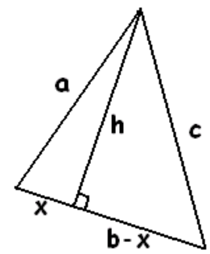
$$\text{area} = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$$

Thus the area A of the 15-20-7 triangle above is

$$A = \frac{1}{4}\sqrt{(42)(2)(28)(12)} = \frac{\sqrt{28224}}{4} = 42\,.$$

Proving Heron's formula is not difficult conceptually. (The algebra required, on the other hand, is a different matter!) Here are two possible approaches:

Proof 1: Draw an altitude and label the lengths x, $b-x$, and h as shown:

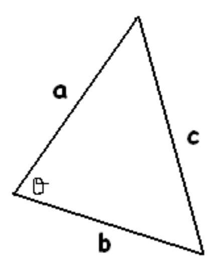


We have

$$(b-x)^2 + h^2 = c^2,$$
$$x^2 + h^2 = a^2.$$

Subtract to obtain a formula for x and substitute to obtain a formula for h. Use $A = \frac{1}{2}bh$ (and three pages of algebra) to obtain Heron's formula.

Proof 2: We have $A = \frac{1}{2}ab\sin\theta$.

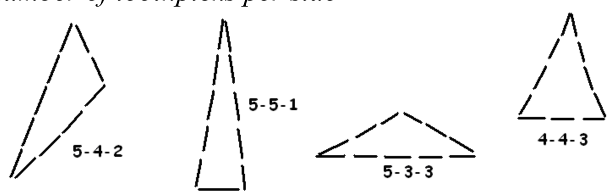


By the law of cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$. Solve for $\sin\theta$ and for $\cos\theta$ and substitute into $\cos^2\theta + \sin^2\theta = 1$.

Out (eventually!) pops Heron's formula.

TOOTHPICK TRIANGLES
Here's a great student activity:

With 11 toothpicks it is possible to make four different triangles of perimeter 11 using a whole number of toothpicks per side



(Why isn't 6-3-2 a valid triangle?)

*The count goes down if one adds another toothpick to the mix: With 12 toothpicks we can make only **three** different triangles: 5-5-2, 5-4-3 and 4-4-4.*

What's going on?

Construct a table showing the number of integer triangles we can make with 1, 2, …, 20 toothpicks. What do you notice about the even and odd entries? Do you see any patterns?

In 2005, high-school students (and some younger) of the St. Mark's Research Group played with this problem and discovered—and proved—the following remarkable formula:

The number of triangles that can be made with N toothpicks is

$\left\langle \frac{N^2}{48} \right\rangle$ *if N is even* and $\left\langle \frac{(N+3)^2}{48} \right\rangle$ *if N is odd.*

(The angled brackets mean "nearest integer to.")

Thus we can make $\left\langle \frac{100^2}{48} \right\rangle = 208$ different triangles with 100 toothpicks!

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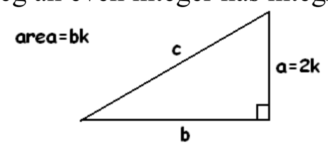
St. Mark's Institute of Mathematics
mathinstitute@stmarksschool.org.

RESEARCH CORNER:
GOING FOR INTEGERS ALL ROUND

The 15-20-7 triangle on the previous page shows that it is possible for a triangle to have integer side lengths and integer area.

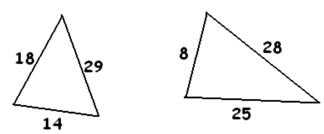
There are many examples. Any right triangle with one leg an even integer has integral area:



But the 15-20-7 triangle has the added property that, not only are its side lengths, perimeter, and area integers, the perimeter and the area have the same value, namely, 42.

Challenge 1: Find another (non-right) integer triangle with perimeter equal to area.

Another idea … It is possible for two different integer triangles to have the same perimeter and the same area? The following two triangles are an example.

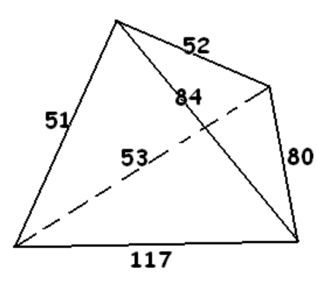


Alas, their areas are not integers.

Challenge 2: Does there exist a pair of integer triangles with the same perimeter and the same integer area?

Challenge 3: Let T(*N*) be the number of integer triangles with perimeter *N* and integral area. Is there a formula for T(*N*) akin to the formula for toothpick triangles?

Just for fun … Here's an integer tetrahedron with each face of integer area and volume also an integer!



INTEGER TRIANGLES

COMMENTARY, SOLUTIONS and THOUGHTS

Each shape in the opening puzzler of the newsletter has the property that the numerical value of its perimeter equals the numerical value of its area. (Is one allowed to say "perimeter equals area"?)

There are only two integer rectangles with this property.

Suppose a rectangle with sides of lengths a and b has area matching perimeter. Then $ab = 2a + 2b$ and so

$$b = \frac{2a}{a-2} = 2 + \frac{4}{a-2}.$$

For the right-hand side to be an integer $a-2$ must be 1, 2, or 4. Thus only the 3×6 rectangle and the 4×4 square have the property.

COMMENT: As observed by high school teacher Michael Ericson, the condition $ab = 2a + 2b$ can be written $\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$. (Read on!)

There are only two integer right triangles with the property:

Suppose a right triangle with legs of integer lengths a and b has area matching perimeter. Then $\frac{1}{2}ab = a + b + \sqrt{a^2 + b^2}$. Squaring

$$a^2 + b^2 = \left(\frac{1}{2}ab - a - b\right)^2 = \frac{1}{4}a^2b^2 + a^2 + b^2 - a^2b - ab^2 + 2ab$$

so

$$a^2b + ab^2 = \frac{1}{4}a^2b^2 + 2ab.$$

Dividing by ab and solving for b gives:

$$b = \frac{4a-8}{a-4} = \frac{4a-16+8}{a-4} = 4 + \frac{8}{a-4}.$$

For the right-hand side to be an integer, a must be 5, 6, 8, or 12. This shows that the 6-8-10 and 5-12-13 triangles are the only integer right triangles with the desired property.

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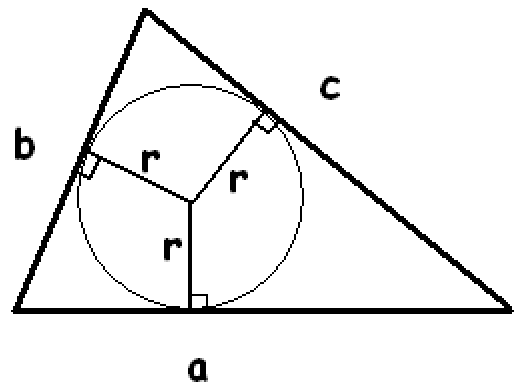
If we remove the requirement that the side lengths of the triangle are integral (and consider arbitrary triangles), we can say:

The perimeter and area of a triangle have the same numerical value if and only if the inradius of the triangle is 2.



Following the notation in the diagram we see that the area A and perimeter P of a triangle satisfy

$$A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}Pr .$$

(The same relation holds for a polygon with sides tangent to a common circle.)

COMMENT: The condition that the inradius r be two can be written $\frac{1}{r} = \frac{1}{2}$.

CURIOUS CHALLENGE:

a) Show that a parallelogram has area equal to perimeter if, and only if, $\frac{1}{h_1} + \frac{1}{h_2} = \frac{1}{2}$ where h_1 and h_2 are its two heights.

b) Show that the area of a triangle equals its perimeter if and only if $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{2}$, where h_1, h_2, and h_3 are the three altitudes of the triangle.

c) Show that an $a \times b \times c$ rectangular prism has volume equal to its surface area if and only if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$.

d) Show that a right circular cylinder has volume equal to surface area if and only if $\frac{1}{r} + \frac{1}{h} = \frac{1}{2}$, where r is the radius of its base and h is its height.

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e) Show that a regular circular cone has volume equal to surface area if and only if $\frac{1}{r}+\frac{1}{b}=\frac{1}{2}$ where r is the radius of its base and b is the perpendicular distance from the center of its base to its lateral surface.

f) Is something going on here? Can we associate with any geometric figure some fundamental set of (perpendicular?) measurements that dictate the properties of its area and perimeter or, in three dimensions, its volume and its surface area? The volume of a sphere equals surface area if and only if $\frac{1}{r}=\frac{1}{3}$. Would it be more appropriate to write $\frac{1}{d}+\frac{1}{d}+\frac{1}{d}=\frac{1}{2}$ where d is the diameter?

See the comment at the end of this essay for a few thoughts on this.

PAIRS OF INTEGER TRIANGLES OF EQUAL AREA AND EQUAL PERIMETER

In the newsletter I presented a pair of integer triangles with the same perimeter and the same area:

The 14-18-29 *and* 8-25-28 *triangles each have perimeter 61 and area* $\frac{5\sqrt{3111}}{4}$.

As is often the case with such problems, if one example exists, infinitely many do! Tyler Jarvis of Brigham Young University pointed me in the right direction to hunt for other solutions.

The challenge is to find integer solutions to

$$a+b+c=d+e+f,$$
$$s(s-a)(s-b)(s-c)=s(s-d)(s-e)(s-f),$$

where $s=\frac{a+b+c}{2}$. (Here a, b, c and d, e, f represent the side-lengths of the two triangles.) Eliminating f and simplifying means we seek integer solutions to

$$\begin{aligned}&(-a+b+c)(a-b+c)(a+b-c)\\&\quad=(a+b+c-2d)(a+b+c-2e)(2d+2e-a-b-c).\end{aligned} \qquad (*)$$

Rational solutions will suffice because a rational solution can be converted to an integer solution by multiplication.

Let S denote the set of all rational points (a,b,c,d,e) that satisfy (*).

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We have within S a host of trivial solutions, namely those that correspond to the two triangles being identical: $d = a$ and $e = b$ (and consequently $f = c$). Let A be any one of them

$$A = (a, b, c, a, b).$$

We also have a particular solution

$$P = (14, 18, 29, 8, 25).$$

Our strategy is to look at the line that connects the point A and the point P and see if it again intersects the set S. For a real number λ, let

$$\begin{aligned} X_\lambda &= \lambda A + (1-\lambda)P \\ &= \big(\lambda a + 14(1-\lambda), \lambda b + 18(1-\lambda), \lambda c + 29(1-\lambda), \lambda a + 8(1-\lambda), \lambda b + 25(1-\lambda)\big). \end{aligned}$$

For X_λ to lie in S its components must satisfy the equation (*). This yields a cubic in λ. We know that $\lambda = 0$ and $\lambda = 1$ are solutions so we can factor λ and $\lambda - 1$ from it to yield a linear equation in λ with rational coefficients. This gives a third rational solution to (*). We need to check that it corresponds to a meaningful geometric solution (namely, the sides of the triangle are all positive and the triangular inequalities hold so that the triangle actually exists!) but the approach gets us beyond the most difficult part of the search: finding examples of numbers that satisfy (*).

With a computer program we can generate many examples of integer triangle pairs with equal areas and equal perimeters. Here are a few:

$10-34-39$ and $19-24-40$
$18-27-30$ and $20-24-31$
$35-57-62$ and $42-47-65$
$40-61-66$ and $46-52-69$
$45-94-94$ and $49-84-100$.

COUNTING INTEGER TRIANGLES

Students of the St. Mark's Institute Research class played with this challenge in 2005 ([TANTON]), as had the 2002 students of the Boston Math Circle ([FOCUS]).

Counting the number of distinct triangles we can form with $n = 1, 2, 3, ...$ toothpicks yields the sequence

0, 0, 1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10, 8, … .

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Denote the n term by $T(n)$. The list is composed of two intertwined copies of the same sequence of numbers 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, … , with one copy shifted three places. That is, it seems that

$$T(2n) = T(2n-3) \text{ for } n > 1.$$

If this is true, we need then only consider triangles with even integer perimeters in our pursuit of a formula for these counts.

Let's take a moment to collate some facts about integer triangles.

1. Three positive integers a, b, and c (written in non-decreasing order) are the side lengths of a triangle if and only if $a+b>c$.

This is the triangle inequality.

2. No integer triangle (or any triangle for that matter) possesses a side of length greater than or equal to half its perimeter.

The remaining two sides would sum to a value less than half the perimeter and this violates Fact 1.

3. No integer triangle with even perimeter has a side of length 1.

This follows from Fact 2.

4. If a, b, and c (written in non-decreasing order) are the sides of an integer triangle of even perimeter, then $a+b>c+1$.

We know $a+b>c$. If $a+b=c+1$, then its perimeter $a+b+c$ would be odd.

5. If a, b, and c are the sides of an integer triangle with even perimeter $2n$, then $a-1$, $b-1$, and $c-1$ are the sides of a valid triangle of (odd) perimeter $2n-3$. Conversely, adding one to the value of each side of a triangle of perimeter $2n-3$ produces a valid triangle of perimeter $2n$.

One checks that the necessary inequalities hold.

Fact 5 shows that we have a match between the triangles of perimeter $2n$ and those of perimeter $2n-3$ and so, indeed, $T(2n) = T(2n-3)$ for all $n > 1$.

In pursuit of a formula for $T(n)$, the 2005 students of the St. Mark's Institute class made the following key discovery:

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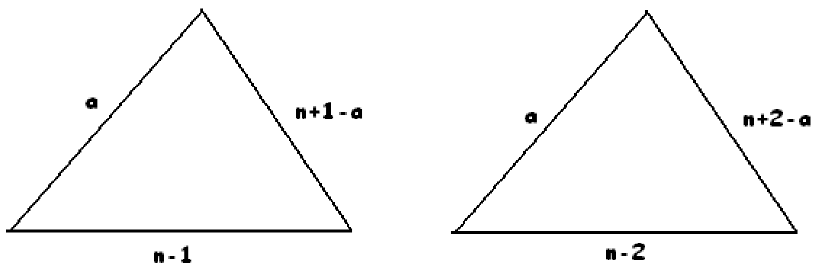
Lemma: *For* N *even with* $N > 12$, $T(N) - T(N-12) = \frac{N}{2} - 3$.

Proof: Write $N = 2n$.

Let $a \le b \le c$ represent the sides of an integer triangle of perimeter $N - 12$. By Fact 2, c is at most $n - 7$. Then $a + 4$, $b + 4$, and $c + 4$ represent the side lengths of a valid triangle of perimeter N with longest side at most $n - 3$. The correspondence

$$(a,b,c) \leftrightarrow (a+4, b+4, c+4)$$

provides a match between the two types of triangles, but it omits the triangles of perimeter N with longest side $n-2$ or $n-1$. How many of these triangles is this correspondence overlooking?



The triangle inequality shows that for a triangle of perimeter $2n$ and longest side $n-1$, the shortest length a of the triangle satisfies $2 \le a \le \frac{n+1}{2}$, and for a triangle of perimeter $2n$ and longest side $n-2$ the shortest side a satisfies $4 \le a \le \frac{n+2}{2}$. There are thus a total of $\left\lfloor \frac{n+1}{2} \right\rfloor - 1 + \left\lfloor \frac{n+2}{2} \right\rfloor - 3 = n - 3$ of these triangles. Consequently $T(N)$ is larger than $T(N-12)$ by $n-3$. This establishes the lemma. □

Define $T(0)$ to be zero. If we write $N = 12k + r$ with $r = 0$, 2, 4, 6, 8 or 10, then

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$$\begin{aligned}T(N) = T(N) - T(N-12) \\ + T(N-12) - T(N-2\cdot 12) \\ + \cdots \\ + T(12+r) - T(r) \\ + T(r)\end{aligned}$$

$$= \frac{12k+r}{2} - 3 \;+\; \frac{12(k-1)+r}{2} - 3 + \;\cdots + \frac{12\cdot 1+r}{2} - 3 + T(r)$$

$$= 6\left(k + (k-1) + \cdots + 1\right) + k\cdot\frac{r}{2} - 3k + T(r)$$

$$= 3k^2 + \frac{1}{2}kr + T(r)$$

$$= \frac{(12k+r)^2}{48} - \frac{r^2}{48} + T(r).$$

We check that $T(r) - \frac{r^2}{48}$ is strictly between $-\frac{1}{2}$ and $\frac{1}{2}$ for each of the six values of r. That is, for each even value N,

\

$$T(N) = \frac{N^2}{48} \pm \varepsilon$$

for some ε less than $1/2$. Thus $T(N) = \left\langle \frac{N^2}{48} \right\rangle$, as claimed in the newsletter. For N odd we have $T(N) = T(N+3) = \left\langle \frac{(N+3)^2}{48} \right\rangle$.

CHALLENGE: Let $S(n)$ be the count of *scalene* integer triangles of perimeter *n*. What do you notice about the sequence of numbers produced?

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FINAL COMMENT ON THE AREA AND PERIMETER OF POLYGONS:

Suppose a polygon of area A and perimeter P has sides of lengths a_1, a_2, $\ldots$, a_k.
Consider the quantity $\frac{P}{2A} = \frac{a_1 + a_2 + \cdots + a_k}{2A}$ (which equals $\frac{1}{2}$ if $P = A$).

Area is the product of two quantities of dimension "length." If there is a natural means to use a sum $a_{i_1} + \cdots + a_{i_n}$ as one of those lengths in the computation of area, then $h = \frac{2A}{a_{i_1} + \cdots + a_{i_n}}$ is a meaningful length in the geometry of the figure and its reciprocal $\frac{1}{h}$ appears in the expression $\frac{P}{2A}$.

For example, for a triangle with sides a, b, and c, its area can be computed as $A = \frac{1}{2}(a+b+c)r$ where r is the inradius of the triangle and

$$\frac{P}{2A} = \frac{1}{r}.$$

Its area can also be computed as $A = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$ (where h_a, h_b and h_c are the altitudes of the triangle) and

$$\frac{P}{2A} = \frac{a}{2A} + \frac{b}{2A} + \frac{c}{2A} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$$

Does thinking along these lines add to or diminish the idea that something interesting is going on in the Curious Challenge presented earlier in this chapter?

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