CHAPTER SEVEN

THE GALLEY METHOD

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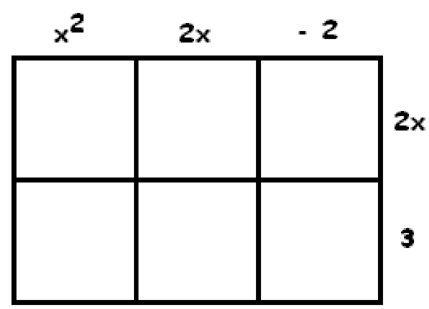
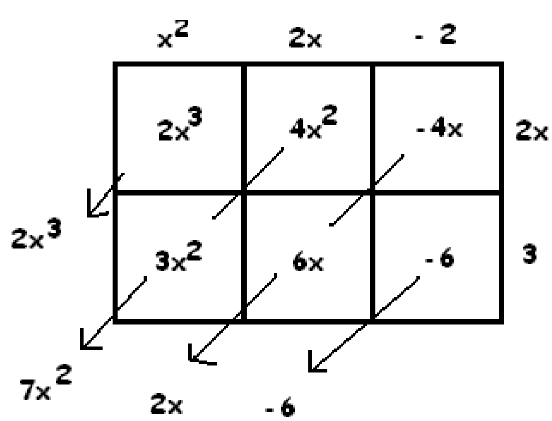
Although the $1 \leftarrow x$ dots-and-boxes model provides a conceptually easy framework for understanding the process of long division of polynimials, it is somewhat tedious to execute. A second disadvantage of this method is its lack of compatibility with the process of long multiplication.

Both of these difficulties are overcome by returning to the area model of multiplication discussed in chapter 1. Let's use it to multiply together x^2+2x-2 and $2x+3$, for instance.

Begin by drawing a rectangle divided into as many columns as there are terms the first polynomial and as many rows as the count of terms of the second polynomial.



To compute the product of the polynomials simply multiply cell by cell and add the results. Adding along the diagonals has the convenience of automatically collecting like powers of *x*.



The answer $2x^3+7x^2+2x-6$ appears.

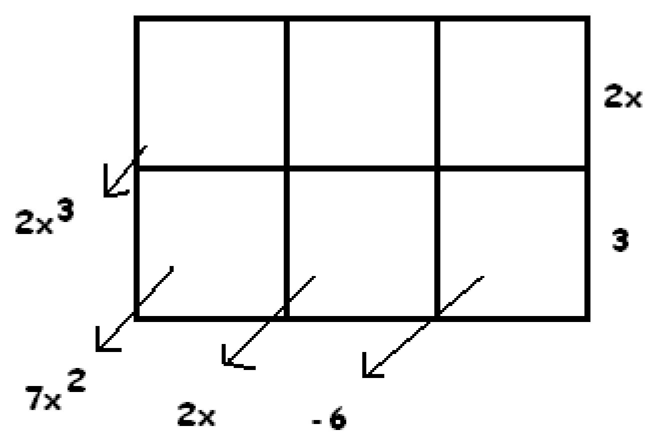
EXERCISE: Compute the following products this way.
(WARNING: Be sure to draw rows and columns for appropriate zero terms!)

a) $\left(3x^5+2x^4-5x^3+4x^2-x+10\right)\left(x^2-3x+4\right)$
b) $\left(x^3-3x+2\right)\left(2x+5\right)$
c) $\left(x^3-x\right)\left(x^3+x^2-x-1\right)$

Because of its similarity with the multiplication method discussed in question 10 of chapter 1, this approach for multiplying polynomials is also called the *galley method*.

A nice feature of the galley method is that we can execute it backwards: If given the answer and one of the original polynomials, it is possible to logically deduce what the entries of the table must be, as well as the identity of the missing polynomial.

EXERCISE: Here is the same table as before with the first polynomial missing and the entries of the table blank.



a) What must the top left cell of the table be?
b) What must be the first term of the missing polynomial?
c) What must the remaining entry of the first column be?
d) Continue in this way to fill in the table and to show that the missing polynomial is indeed x^2+2x-2.

The above example has, in effect, computed $\dfrac{2x^3+7x^2+2x-6}{2x+3}$ for us.

EXERCISE: Solve the following division problems using the reverse galley method. (To do this, one must first determine what size grid to draw. The number of rows is always clear. The number of columns requires a little thought.)

a) $\dfrac{2x^2+7x+6}{x+2}$

b) $\dfrac{x^4+2x^3+4x^2+6x+3}{x^2+3}$

c) $\dfrac{x^6-x^5+x^4-2x^3+6x^2-6x+4}{x^4-2x+4}$

d) $\dfrac{x^8-1}{x+1}$

COMMENT: One can deduce the number of columns one needs in a division problem by taking note of the highest powers of x that appear in the problem. Note, for example, multiplying $4x^4+2x^3-x^2-x+3$ and x^2-2x+1 together produces a polynomial with highest power x^6. On the other hand, dividing $4x^6-6x^5-x^4+3x^3+4x^2-7x+3$ by x^2-2x+1 must yield an answer with highest power x^4. (Its answer times x^2-2x+1 must produce $4x^6-6x^5-x^4+3x^3+4x^2-7x+3$.) Thus we can deduce that the division problem:

$$\frac{4x^6-6x^5-x^4+3x^3+4x^2-7x+3}{x^2-2x+1}$$

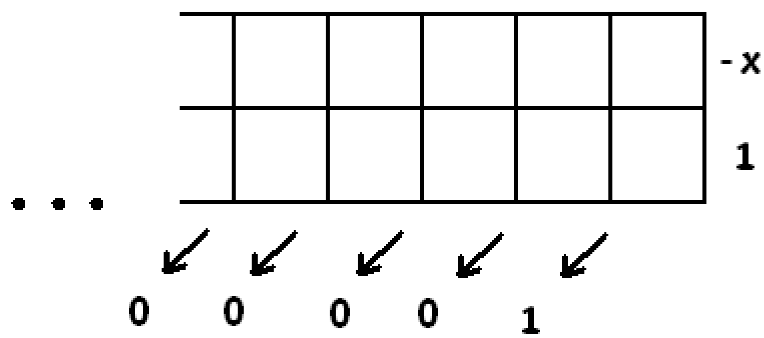
requires a table with five columns, one for each of the powers 1, x, x^2, x^3, and x^4.

EXERCISE: Compute $\dfrac{4x^6-6x^5-x^4+3x^3+4x^2-7x+3}{x^2-2x+1}$, but use a table with three rows and EIGHT columns. Verify that misjudging the number of columns to the excess offers no cause for concern.

One can go further with the galley method.

EXERCISE:

a) Complete the following table, with infinitely many columns to the left, to evaluate $\frac{1}{1-x}$.



b) Compute $\frac{1}{1+x}$ via the reverse galley method.

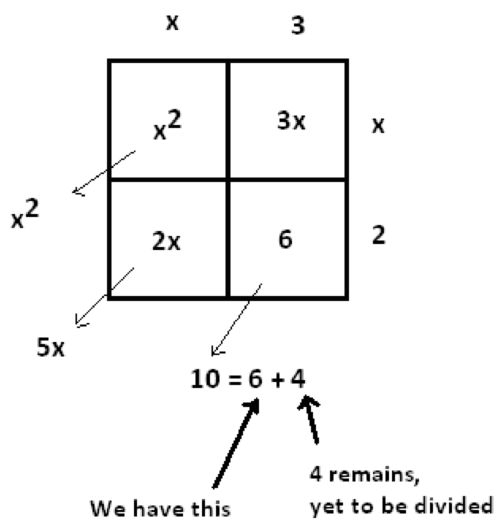
c) Compute $\frac{1}{1-x^2}$ via the reverse galley method.

d) Compute $\frac{1}{1-x-x^2}$ via the reverse galley method. What famous sequence of numbers do you see appearing?

REMAINDERS

The reverse galley method shows that $\frac{x^2+5x+6}{x+2}$ equals $x+3$. (Check this!)

Consequently, dividing $x^2+5x+10$ by $x+2$ must leave a remainder of 4. Let's see how this appears in the table as we work through the problem:



The table does indeed show:

$$\frac{x^2+5x+10}{x+2}=x+3+\frac{4}{x+2}$$

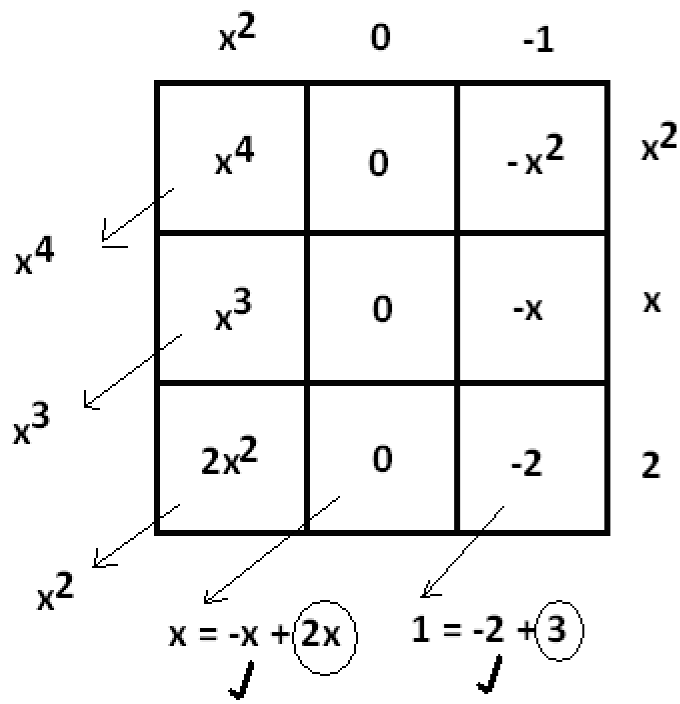
(Multiply through by $x+2$ to check this.)

As another example, consider

$$\frac{x^4+x^3+x^2+x+1}{x^2+x+2}$$

Perhaps try it before reading on.

We have:



This reads:

$$\frac{x^4 + x^3 + x^2 + x + 1}{x^2 + x + 2} = x^2 - 1 + \frac{2x + 3}{x^2 + x + 2}$$

EXERCISE: Compute

$$\frac{x^5 - x^3 + x + 1}{x^4 - 2}$$

EXERCISES

Question 1: Compute the following via the reverse galley method:

a) $\dfrac{2x^5 - 6x^4 + 7x^3 + 2x^2 - 11x - 3}{x^3 - 2x^2 + 3x + 1}$

b) $\dfrac{x^6 - 64}{x - 2}$

c) $\dfrac{2x^8 - 3x^5 + 3x^4 + x^2 - 3x}{x^3 - 1}$

Question 2:

a) Apply the reverse galley method to the problem $\dfrac{x^2 - 6x + 11}{x - 2}$ to show that $\dfrac{x^2 - 6x + 11}{x - 2} = x - 4 + \dfrac{3}{x - 2}$.

b) Evaluate $\dfrac{x^2 + x + 1}{x - 5}$

c) Evaluate $\dfrac{2x^2 + 15x - 63}{x + 10}$

d) Explain the following:

A quadratic $ax^2 + bx + c$ *divided by a linear term* $x - h$ *is sure to be of the form:*

$$\frac{ax^2 + bx + c}{x - h} = Ax + B + \frac{d}{x - h}$$

with A, B and d numbers.

In fact, go further and use the reverse galley method to show that:

$$A = a$$
$$B = ah + b$$
$$d = (ah + b)h + c$$

Notice that $d = ah^2 + bh + c$, the value of the quadratic at $x = h$.

[**COMMENT**: One can also see this (with less work!) by taking the equation $\frac{ax^2 + bx + c}{x - h} = Ax + B + \frac{d}{x - h}$ and multiplying through by $x - h$. This gives:

$$ax^2 + bx + c = (Ax + B)(x - h) + d$$

Now substitute in $x = h$ to see that $d = ah^2 + bh + c$.]

e) Explain the following:

If a quadratic $ax^2 + bx + c$ has value 0 at $x = h$, then $x - h$ is a factor of the quadratic.

COMMENT: This result is a special case of the remainder theorem which we shall discuss in chapter 27.

Question 3: Compute:

a) $\frac{x^4 + 4x^3 + 5x^2 + 6x - 2}{x^2 + x + 1}$

b) $\frac{x^5 + x^4 + x^3 + 1}{2x^3 + 3}$

Question 4:

a) Use the reverse galley method evaluate $\frac{x^3+8}{x+2}$.

b) Making this a little more abstract, evaluate $\frac{x^3+a^3}{x+a}$.

c) Use the reverse galley method to show that x^5+a^5 is divisible by $x+a$.

d) Explain why, for *n* odd, x^n+a^n is always multiple of $x+a$.

e) Show that $7^{50}+22^{25}$ is divisible by 71.