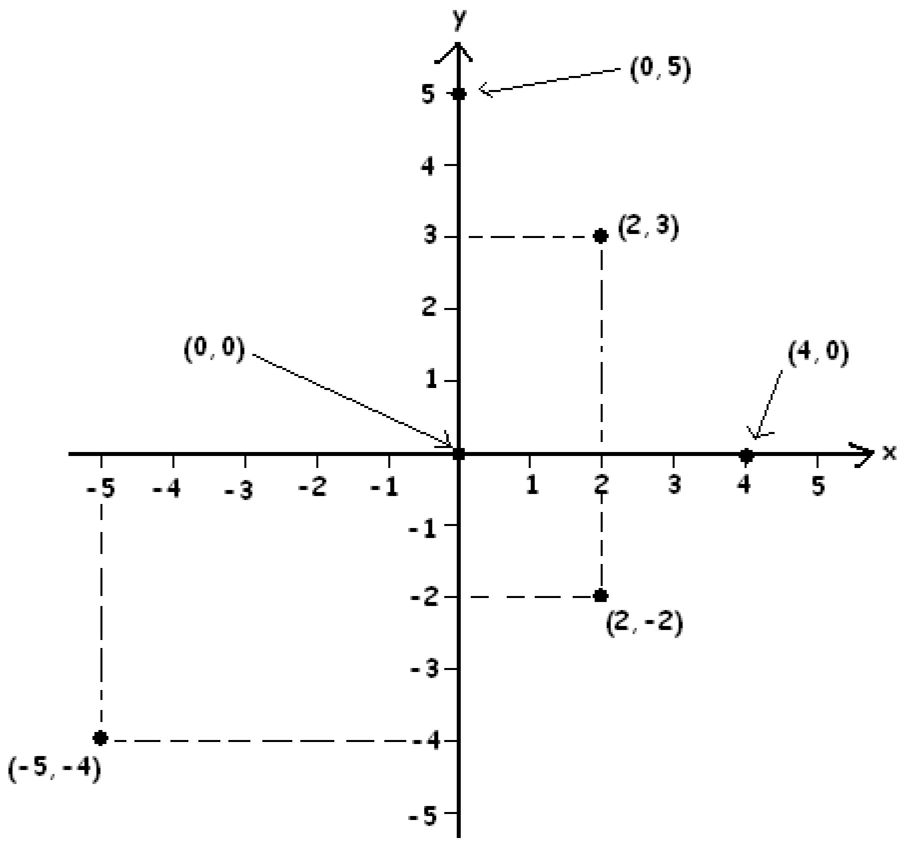
UNIT SIX

MODERN APPLICATIONS OF PYTHAGORAS'S THEOREM

COORDINATE GEOMETRY

Geometry, as presented by Euclid, stood on its own for almost two millennia. But in the 1600s western European scholars discovered a way to unite the subject with the power of algebra through the newly discovered notion of a coordinate system.

One draws two number lines set at 90^o to one another and uses these "axes" to associate to each point in the plane a pair of numbers.



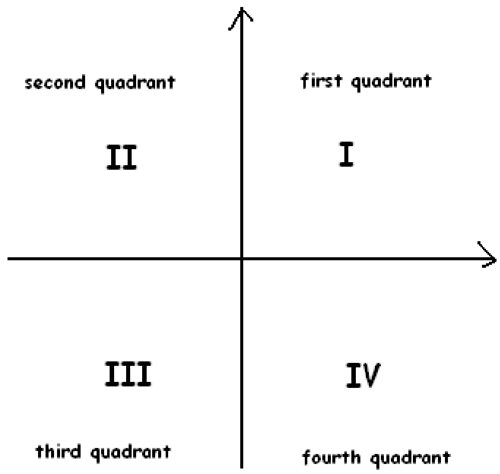
It is conventional to call the horizontal axis, with positive numbers to the right, the *x*-axis, and the vertical axis, with the positive numbers upwards, the *y*-axis.

Each point (x, y) is listed with the "*x*-coordinate" first and the "*y*-coordinate" second.

SOME JARGON:

The point $(0,0)$ is called the origin.

The two axes divide the plane into four quadrants.



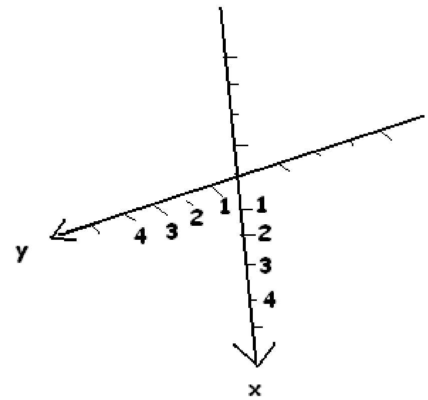
Comment: See the unit on trigonometry for an explanation as to why mathematicians place the quadrants in this counter-clockwise order.

HISTORICAL NOTE

Many people say that French philosopher and mathematician Renè Descartes (1596-1650) invented the concept of a coordinate system. This is not quite true: Descartes never thought to explicitly draw a vertical axis. He did, however, draw pictures of "graphs" that seemed to use a second axis paired with a horizontal number line, but he never held that second axis at a fixed 90^o to the horizontal: the angle of the second axis varied from problem to problem. Descartes, however, did manage to solve a large number of challenging questions using coordinate systems. For this reason, he is dubbed the "father of coordinate geometry." (And the coordinate system we use is sometimes called a Cartesian coordinate system to honor his name.)

No one knows when or how the idea of using a fixed vertical axis came into place. It seems to be an idea that simply evolved during the 1700s.

Another Comment: There really is nothing special about the way we place the *x*- and *y*-axes. We could, if we so decided, use axes as follows at an angle of 83^o oriented as shown:

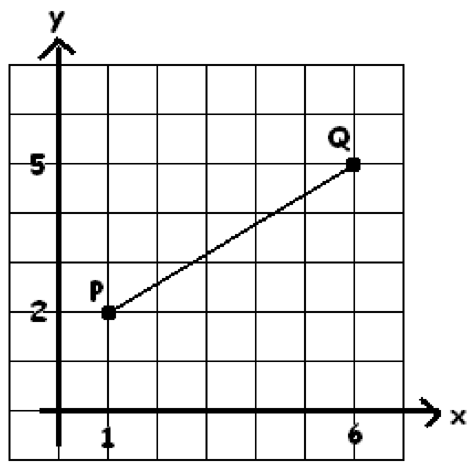


It is a <u>convention</u> that we use vertical and horizontal axes with positive numbers to the right and to the top. It is also a <u>convention</u> that we choose to read the coordinates of a point $(2,3)$ with the first number "2" the *x*-coordinate and the second number "3" the *y*-coordinate.

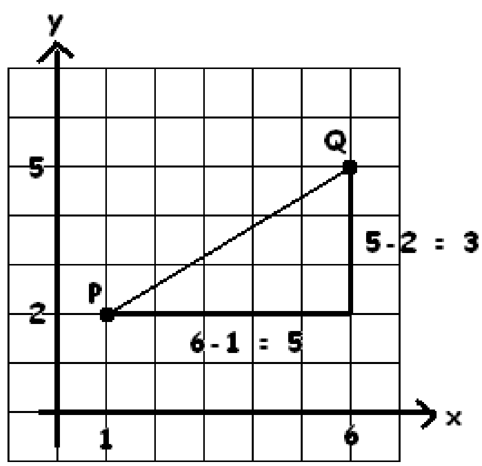
However, it turns out to be exceptionally convenient to work with axes that lie at 90^o to one another. As we shall next see, this allows us to make use of Pythagoras's theorem in an easy and natural way.

THE DISTANCE FORMULA:

Here are two points P and Q on the coordinate plane. What is the distance between them?



The advantage to using a pair of horizontal and vertical axis is that we are able to draw right triangles at different locations with ease. In this example, we can see that the line segment $\overline{PQ}$ is the hypotenuse of a right triangles with legs of lengths $6-1=5$ and $5-2=3$.



Thus, by Pythagoras theorem:

$$\text{distance } PQ = \sqrt{\left(\text{difference in x-values}\right)^2 + \left(\text{difference in y-values}\right)^2}$$

$$= \sqrt{\left(6-1\right)^2 + \left(5-2\right)^2} = \sqrt{5^2+3^2} = \sqrt{34}$$

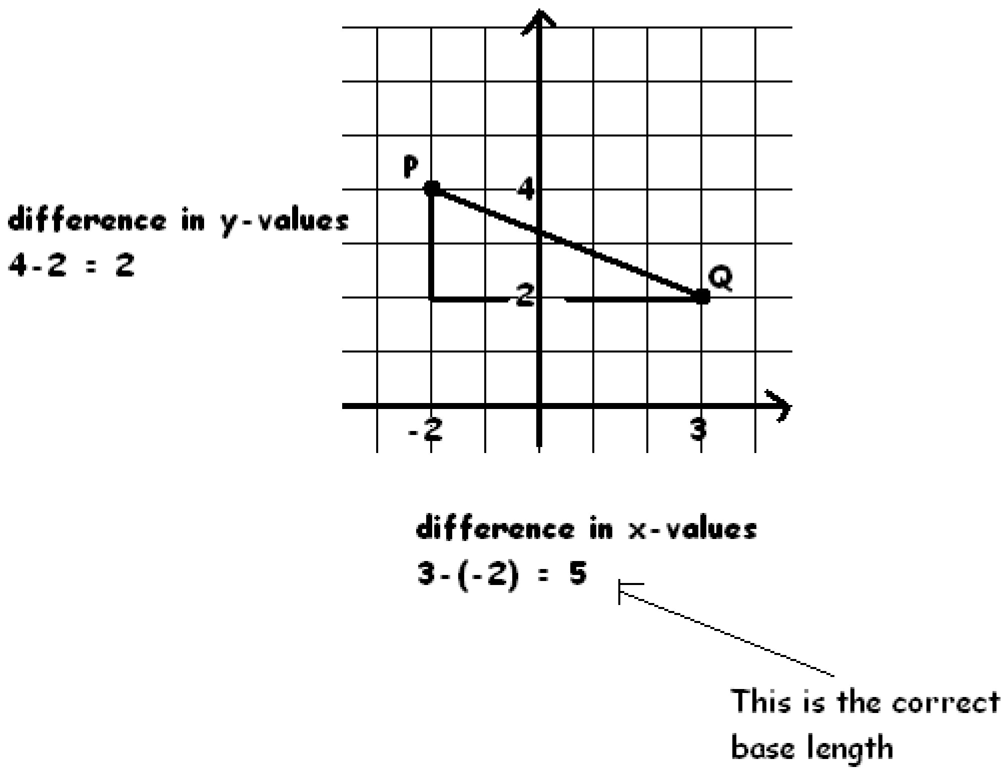
Question: Does the formula:

$$\text{distance } = \sqrt{\left(\text{difference in x-values}\right)^2 + \left(\text{difference in y-values}\right)^2}$$

always work? What if the coordinates of some points are negative?

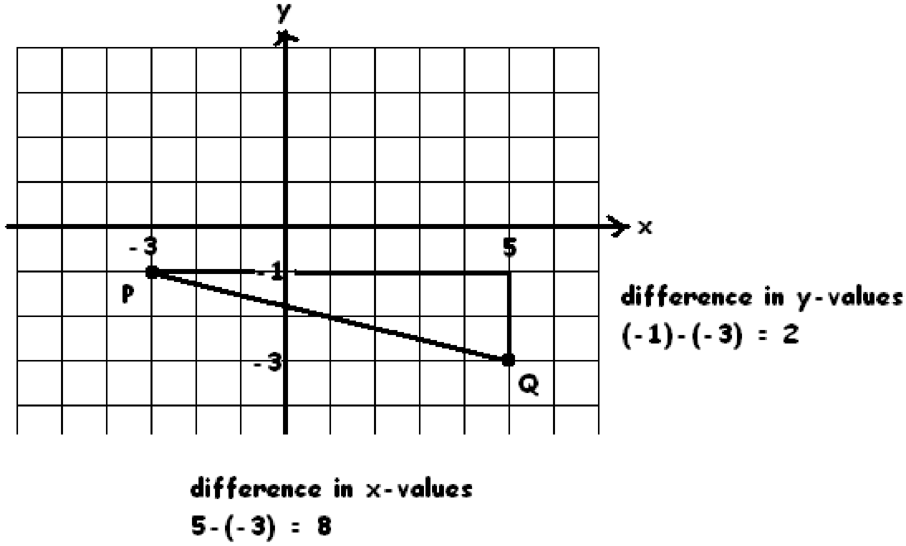
Let's examine a few examples to see that this formula is indeed valid in all situations.

EXAMPLE:



distance = $\sqrt{\left(3-(-2)\right)^2+(4-2)^2} = \sqrt{5^2+2^2} = \sqrt{29}$

EXAMPLE:



Distance = $\sqrt{8^2+2^2}=\sqrt{68}$

In general, given two points $P=(x_1,y_1)$ and $Q=(x_2,y_2)$, one can always draw a right triangle with hypotenuse $\overline{PQ}$.

The horizontal leg of the triangle has length given by the difference of the *x*-coordinates (and this is true even with negative values present) and the vertical leg of the triangle has length given by the difference of the *y*-coordinates. Thus by Pythagoras's theorem we have:

DISTANCE FORMULA: The distance between two points $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ is given by:

$$\text{PQ} = \sqrt{(\text{difference in x-values})^2 + (\text{difference in y-values})^2}$$

Comment: This formula can be written: $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

Note: The distance formula has the advantage that it does not matter in which order one decides to subtract the two x-coordinates or the two y-coordinates.

e.g. The distance between $P=(2,3)$ and $Q=(5,8)$ can be computed as:

$$PQ=\sqrt{(5-2)^2+(8-3)^2}=\sqrt{3^2+5^2}=\sqrt{34}$$

or as

$$PQ=\sqrt{(2-5)^2+(3-8)^2}=\sqrt{(-3)^2+(-5)^2}=\sqrt{34}$$

Reason: Even though x_2-x_1 and x_1-x_2 are not the same (in fact, one is the negative of the other), their squares are the same.

$$(x_2-x_1)^2=(x_1-x_2)^2.$$

PRACTICE EXERCISE: Find the distances between the following pairs of points:

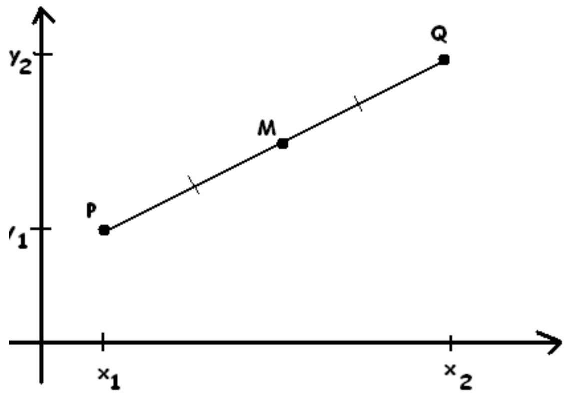
a) $A=(2,10)$ $B=(2,8)$ (Does the answer surprise you?)
b) $C=(-7,4)$ $D=(-3,1)$
c) $P=(-3,3)$ $Q=(8,-2)$

PRACTICE EXERCISE: Let $A=(3,4)$, $B=(5,10)$ and $R=(4,7)$.

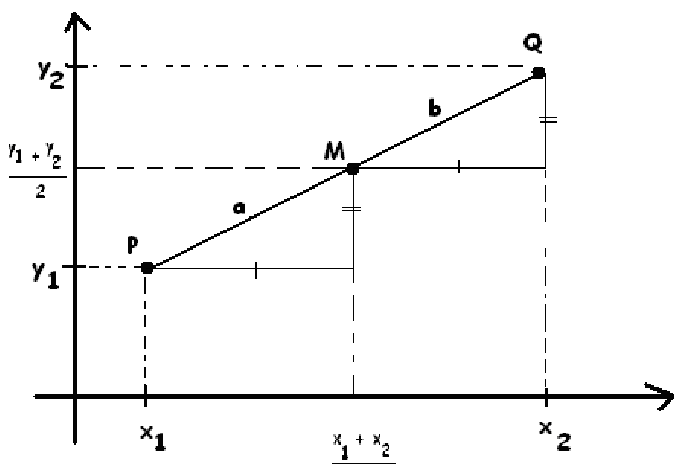
a) Show that AR=RB
b) Suppose $S=(1,k)$ is another point with AS = SB. What is the value of k?

MIDPOINT FORMULA:

Suppose we wish to find the coordinates of the midpoint *M* of a line segment $\overline{PQ}$.



One might guess that *M* has *x*-coordinate given by the value exactly half way between x_1 and x_2, namely, $\frac{x_1 + x_2}{2}$; and that that *y*-coordinate of *M* might be the value exactly half way between y_1 and y_2, namely, $\frac{y_1 + y_2}{2}$. This gives two right triangles with bases of equal lengths and heights of equal length. By Pythagoras's theorem it seems that, in the diagram below, we would indeed have *a* equal to *b* making *M* the midpoint of $\overline{PQ}$.



But there is a problem. How do we know for sure that this point *M* with coordinates given as the average of the *x* values and the average of the *y* values actually lies on the line $\overline{PQ}$? As we have seen, that PM = MQ doesn't automatically imply that *M* is the midpoint of $\overline{PQ}$!

It turns out that all is okay and that M given by this approach is indeed the midpoint of the line segment. Proving this, however, is quite tricky. We leave it as an optional exercise for the brave. (Though, in unit 16, we'll see a quick geometric way to prove the formula true.)

MIDPOINT FORMULA:
The midpoint of the line segment connecting

$P = (x_1, y_1)$
and $Q = (x_2, y_2)$

is $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$

EXAMPLE: Find the midpoint M of the line segment connecting $A = (-1, 3)$ to $B = (2, 8)$.

Show that $AM = MB$ to verify that M is indeed equidistant from A and B. Also verify that $AM + MB$ equals AB to show that M does indeed lie on $\overline{AB}$.

Answer: The midpoint is given by: $M = \left(\dfrac{-1+2}{2}, \dfrac{3+8}{2} \right) = \left(\dfrac{1}{2}, \dfrac{11}{2} \right)$.

The distance between $\left(\dfrac{1}{2}, \dfrac{11}{2} \right)$ and $(-1, 3)$ is:

$$AM = \sqrt{\left(-1 - \frac{1}{2}\right)^2 + \left(3 - \frac{11}{2}\right)^2} = \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{-5}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{25}{4}} = \sqrt{\frac{34}{4}} = \frac{\sqrt{34}}{2}$$

The distance between $\left(\dfrac{1}{2}, \dfrac{11}{2} \right)$ and $(2, 8)$ is:

$$MB = \sqrt{\left(2 - \frac{1}{2}\right)^2 + \left(8 - \frac{11}{2}\right)^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{34}{4}} = \frac{\sqrt{34}}{2}$$

These distances are indeed the same.

Also, $AB = \sqrt{3^2 + 5^2} = \sqrt{34}$ and so $AM + MB$ does equal AB. □

PRACTICE EXERCISE:

a) Suppose $A = (3,4)$ and $T = (6,11)$. Find the coordinates of a point B so that T is the midpoint of $\overline{AB}$.

b) Suppose $P = (2k,3m)$ and $R = (5k,-m)$. Find the coordinates of a point S so that R is the midpoint of $\overline{PS}$.

SUMMARY OF KEY RESULTS

DISTANCE FORMULA: The distance between two points $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ is given by:

$$\text{PQ} = \sqrt{(\text{difference in x-values})^2 + (\text{difference in y-values})^2}$$

ADVICE: Don't memorize this formula! If you realize that this is just Pythagoras's theorem you can easily reconstruct it when you need it.

MIDPOINT FORMULA:
The midpoint of the line segment connecting $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ is:

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

ADVICE: Don't memorize this formula! Just realize that the midpoint has coordinates that are the average of the x and y coordinates of the original points.

EXERCISES:

Question 1: Identify the quadrant or other specific location of:

a) a point with negative x-coordinate and positive y-coordinate
b) a point with y-coordinate zero
c) a point with both coordinates negative

Question 2: What is the name of the point where the coordinate axes intersect?

Question 3: Find the distances between the following pairs of points. (Leave your answers in terms of square roots.)

a) (11,0) and (5,9)
b) (4, 7) and (7,4)
c) (0, 0) and (-2, -5)
d) (4, 400) and (-3, 424)
e) (-2, 11) and (6, 15)

Question 4: Find the coordinates of the midpoint of $\overline{PQ}$ if

a) P = (5,6) and Q = (11, 4)
b) P = (-3, 17) and Q = (15, 37)
c) P = (-12, 6) and Q = (0, 13)
d) P = (a, 0) and Q = (-a, b)

Question 5: a) $A=(a,7)$ and $M=(2a,2)$ and M is the midpoint of $\overline{AB}$, what are the coordinates of B?
b) Find a value for x so that $M=(2x+2,\ 9)$ is the midpoint of $\overline{PQ}$ with $P=(x,x+3)$ and $Q=(3x+4,2x)$.

Question 6: A triangle ABC has vertices $A=(-1,2)$, $B=(1,6)$ and $C=(3,0)$. Find the coordinates of the midpoint M of $\overline{BC}$ and the length of the line segment $\overline{MA}$.

(A line connecting one vertex of a triangle to the midpoint of the opposite side is called a <u>median</u> of the triangle.)

Question 7: A point P is said to be <u>equidistant</u> from two points A and B if $AP = PB$.

Let A = (-2, 3) and B = (2, -1).

a) Show that the point P = (4,5) is equidistant from A and B.
b) Is the point Q = (-4, -5) equidistant from A and B?
c) Is the point R = (-10, -9) equidistant from A and B?
d) **CHALLENGE**: Show that every point of the form $(k, k+1)$ is equidistant from A and B.

Question 8: Find a point with *y*-coordinate 7 that is 5 units away from (2,3).

COMMENT: There are two possible answers. Can you find them both?

Question 9: a) Find an equation that must be true for the numbers *x* and *y* if the point $A = (x, y)$ is a distance of 6 units from $B = (2,9)$.

b) When Jinny answered this question she came up with the formula:
$(x-2)^2 + (y-9)^2 = 36$. Is this correct?

Question 10: Let F = (3,4) and G = (13, -6).

a) Find the coordinates of the point half-way along $\overline{FG}$. (That is, find the midpoint of $\overline{FG}$.)
b) Find the coordinates of the points that are one-quarter and three-quarters along $\overline{FG}$.

Question 11: A triangle is said to be <u>isosceles</u> if at least two of its sides are the same length.

Let A = (0,2); B = (12, 10) ; C = (4, -2).

a) Use the distance formula to show that the triangle formed by the points A, B and C is isosceles.
b) Compute the midpoints of the three sides of this triangle.
c) Is the triangle formed by the midpoints also isosceles?

Question 12: Is there a point with the same *x*- and *y*- coordinates that is a distance 10 from the point (3,5)? If so, what is it? (And is there more than one point with this property?)

Question 13: (OPTIONAL)

The goal of this exercise is to prove the midpoint formula:

MIDPOINT FORMULA:
The midpoint of the line segment connecting

$$P = (x_1, y_1)$$

and $Q = (x_2, y_2)$

is $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

<u>Warning</u>: The algebra is hefty!

First we need to show that PM and MQ have the same value.

This requires using the distance formula.

First note that:

$$\frac{x_1 + x_2}{2} - x_1 = \frac{1}{2}x_1 + \frac{1}{2}x_2 - x_1 = \frac{1}{2}x_2 - \frac{1}{2}x_1$$

$$x_2 - \frac{x_1 + x_2}{2} = x_2 - \frac{1}{2}x_1 - \frac{1}{2}x_2 = \frac{1}{2}x_2 - \frac{1}{2}x_1$$

with similar equations for the *y*-values.

a) Show that $PM = \sqrt{\left(\frac{1}{2}x_2 - \frac{1}{2}x_1\right)^2 + \left(\frac{1}{2}y_2 - \frac{1}{2}y_1\right)^2}$

b) Show that MQ is the same!

Second we need to show that *M* is actually on the line segment $\overline{PQ}$.

We'll do this by showing that $PM + MQ = PQ$, which requires more algebra!

We have:

$$PM = \sqrt{\left(\frac{1}{2}x_2 - \frac{1}{2}x_1\right)^2 + \left(\frac{1}{2}y_2 - \frac{1}{2}y_1\right)^2}$$

c) Show that this can be rewritten as: $PM = \sqrt{\frac{1}{4}(x_2 - x_1)^2 + \frac{1}{4}(y_2 - y_1)^2}$

d) Show that this can be rewritten as: $PM = \frac{1}{2}\sqrt{\left(x_2 - x_1\right)^2 + (y_2 - y_1)^2}$

Notice that this says that PM is half the value of PQ (as we would have hoped!) We know that *MQ* has the same value of *PM*, so *MQ* is also half of *PQ*.

e) Explain now why $PM + MQ = PQ$ must indeed be true.

We're done!