CURRICULUM INSPIRATIONS

TIDBITS FOR THE CLASSROOM INSPIRED BY MAA AMERICAN MATHEMATICS COMPETITIONS

MATHEMATICAL ASSOCIATION OF AMERICA

ESSAY NUMBER: 8

Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty. (From amc.maa.org.)

In this eighth essay of the Curriculum Inspirations series we look to help with test-focussed questions. Standardized assessment is a reality of the student experience and tailored problem-solving steps can be valuable in this setting too. Our feature strategy this time is:

> *Can we second-guess the author of the question?*

Curriculum Commentary: There are parts of the standard curriculum that are contrived. For example, we teach students to factor quadratics for a whole slew of examples designed specifically to factor. But reality is that most quadratics don't factor nicely at all! (For example, when confronted with $2x^2 - 4x + 1$ do naturally think $x - 2 \pm \frac{1}{\sqrt{2}}$?) We give factoring the weight of being a useful additional technique for solving equations. Hmm.

We might also give the impression that many features are of the curriculum have direct real-world application. Many algebra texts have exercises that begin:

> *A company has determined that its profits are modeled well by the following quadratic formula. ...*

Of course most companies do not have a formula of any kind for their profits (and in the real world finding formulas for phenomena is often nigh-on impossible). The "neatness" of this profit scenario is contrived.

Nonetheless, pre-set, unrealistic scenarios often have a necessary role in the pedagogical development of the mathematics. And we can be honest about this! Students are comfortable with frank and open conversation.

For example, we can say:

Most polynomials don't factor. But test makers are going to want you to recognize that $x^2 - 5x + 6$ *is the same as* $(x-2)(x-3)$, *even though that we all know the story of symmetry is the best way to solve quadratics.*

OR…

This question about a bolt manufacturer using quadratics to make production decisions is silly. Let's rewrite the question and have her manufacture buckets of gloop instead. Let's enjoy the silliness!

OR… (with less snide an attitude!)

Factoring in and of itself is not the goal. The ideas we are developing by seeing patterns, recognizing structure, and so on, while practicing factoring are important. We are setting the stage for bigger thinking skills.

Students like to be let in on the reality of matters. They too see though the contrived nature of things, when they occur, and appreciate an adult response to it. It can be a bonding experience for the class, with you included, to have a shared understanding and acknowledgment of the forced devices used by textbook authors and test-makers.

Our tidbit today is **query 5** from the **2011 AMC 10b** experience. We will look at this question and attempt read the mind of its author. We acknowledge that most real-world problems do not give us this opportunity, that it is usually very hard to second-guess nature and the mathematical universe in this way. We understand that we are having fun engaging in a contrived experience and are simply enjoying some mathematics.

The question:

> In multiplying two positive integers a and b, Ron reversed the digits of the two-digit number a. His erroneous product was 161. What is the correct value of the product of a and b?

Solving the Question:
As per usual we start with step 1 of the problem solving journey:

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1: Read the question, have an
emotional reaction to it,
take a deep breath, and then
reread the question.
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There seems to be little to go on in this question. Vaguely … it is about multiplying numbers together to get an incorrect and then a correct answer. The information provided feels loose.

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2: Understand the question.
Understand the different
components of the question.
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Okay, we have two numbers, one called a and once called b. We are told that a is a two-digit number. (Does that mean that b isn't?) The digits of a got reversed somehow.

There is something about 161 being the wrong answer. We are asked to get the right one.

161 *is the wrong answer to what?*

It is the wrong answer to the product $a \times b$.

Now what?

We seemed to have covered all the key parts of the question. Even if we go back and underline all the verbs and the key words in the question statement (often a good test-question answering device) there seems to be nothing significant we've missed. We could, I suppose, add: 161 *is the wrong answer to the product* $a \times b$. *It is:*

$$\textit{(reverse of } a\textit{)} \times b = 161.$$

But we do need something more.

Let's employ our feature strategy:

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3: Can we second-guess the
author of the question?
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The number 161 is the only piece of mathematical detail offered in the question. Why the number 161? Surely the author didn't choose this number out of thin air?

What do we know about the number 161?

It is odd. (Always good to state the basics!)

It feels as though it might be prime. Is it?

Actually, would the author want it to be prime? After all, we need

$$\textit{(reverse of } a\textit{)} \times b = 161.$$

and this is saying we need 161 to factor.

Okay, so I bet 161 factors. Ugh!

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Avoid hard work if you can.
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161 is odd, so it is not divisible by 2. Nor then is it divisible by 4 or 6 or 8 or …. This leaves odd factors to consider.

Is 161 *divisible by three?*
I know there is some "divisibility rule" for the number three, but my mind has gone blank. Is there an easy way to check whether or not 161 is a multiple of three?

Well, 150 is a multiple of three near to 161 ($150 = 3 \times 50$) and $161 = 150 + 11$. Since 11 is <u>not</u> a multiple of three, 161 can't be either. (That was smart!)

Is 161 *divisible by five?*
I remember a divisibility rule for five, but writing $161 = 150 + 11$ also makes it clear that 161 is not a multiple of five.

[We discuss divisibility rules at the end of this essay.]

Is 161 *divisible by seven?*
I wonder if there is a divisibility rule for seven? I don't need one, because I can think of a multiple of seven pretty close to 161, namely, 140, which is 20×7. We can write:

$$161 = 140 + 21.$$

Bingo! We see that 161 is a multiple of seven:

$$\begin{aligned} 161 &= 140 + 21 \\ &= 20 \times 7 + 3 \times 7 \\ &= (20 + 3) \times 7 \end{aligned}$$

We have: $161 = 23 \times 7$.

(SO… Factoring for the purposes of arithmetic is meaningful and helpful. It is a pity that number theory is not part of the standard high-school curriculum!)

Since the numbers 7 and 23 are prime, this is the only factorization of the number 161. (Do we teach the uniqueness of prime factorizations in school? See "Are factor trees unique?" at www.jamestanon.com/?p=680 to see why this is a meaningful question!)

Seeing $161 = 23 \times 7$ makes the entire question fall into place! From:

$$(\textit{reverse of } a) \times b = 161$$

we must have that

$$\textit{reverse of } a = 23$$
$$b = 7$$

(recall a is a two-digit number) and so:

$$a = 32$$
$$b = 7$$

and the correct product is:

$$a \times b = 32 \times 7$$

The easiest way for me to do arithmetic is to expand products:

$$32 \times 7 = (30 + 2) \times 7 = 210 + 14 = 224.$$

And this is the answer to our question 10b 5!

DECONSTRUCTING THE PROBLEM:
Was there indeed something special about the number 161 for the question? This is a good query to put to students. A helpful prompt for discussion might be:

Could this question have worked just as well using the number 168 *(which factors as* 24×7 *) in place of* 161*?*

One can challenge students to devise their own version of this problem where a is a three-digit number, b is a two-digit number, and the erroneous product *(reverse of* a*)* $\times\, b$ is supplied.

Puzzle: In the question "*reverse of* a" has value 23, a prime number, but a itself is 32, not prime. Is there a two digit number such that it and its reverse are both prime? Is there a three-digit number such that it and all numbers that result from rearranging its digits are prime? A four-digit number like this?

A small point … So much of the standard curriculum is focused on content, content, content. It is easy to side-step clever ideas to help students rely on their wits and engage in self-created clever thinking. Consider the following series of questions:

Is 1938 *a multiple of* 19*?*
Is 203 *divisible by* 7*?*
What is the remainder in dividing 203 *by* 98*?*
What is $17\frac{1}{2}\%$ *of* 80*?*

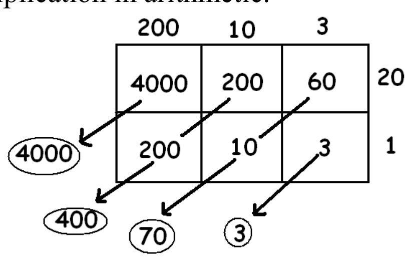
Do you personally see the answers: YES, YES, $2 + 2 + 3 = 7$ and $8 + 4 + 2 = 14$?

What lovely sophisticated thinking small arithmetic questions can foster!

CURRICULUM CONNECTION:
At face value it seems that our MAA AMC 2011 problem is irrelevant to the high-school curriculum. It is about the product and factorization of whole numbers, a subject of the purview to the elementary grades.

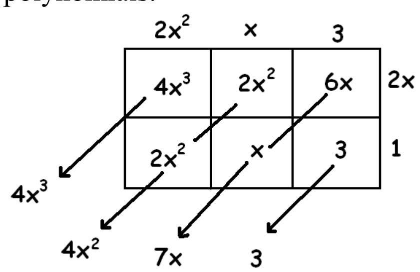
But CCSS-M standard **A-APR-1: Perform Arithmetic Operations on Polynomials** explicitly asks educators to help students understand that "polynomials form a system analogous to the integers." We can take the factoring and expanding of whole numbers and make the connection direct!

Here is a natural way to perform long-multiplication in arithmetic.



We see $213 \times 21 = 4473$.

Here is a natural way to multiply two polynomials:



We see

$$\left(2x^2 + x + 3\right)\left(2x + 1\right) = 4x^3 + 4x^2 + 7x + 3$$

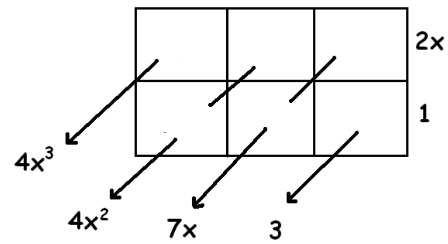
They are no different!

QUERY:
Forget that you see the table above.

Suppose you were given the answer $4x^3 + 4x^2 + 7x + 3$ and the factor $2x + 1$. Could you fill in blank spots of the table in the next column and deduce what the other factor must be?

That is, can you use this blank table to compute

$$\frac{4x^3 + 4x^2 + 7x + 3}{2x + 1} ?$$



We can teach the division of polynomials in a way that is natural to the arithmetic we learn in the early grades. (See the sample chapter from *THINKINGMATHEMATICS! Vol 1: Arithmetic=Gateway to All*, at www.jamestanton.com/?p=1290 for the details of this approach.)

Going further …

In the high-school curriculum we examine factored formulas such as:

$$x^3 - 1 = \left(x - 1\right)\left(x^2 + x + 1\right)$$

But we tend to forgot in an algebra class that x can actually be a number!

Is $2^{30} - 1$ prime?

Answer: $2^{30} - 1 = \left(2^{10}\right)^3 - 1 = 1024^3 - 1$.

Put $x = 1024$ into the formula for $x^3 - 1$ and we see that

$$2^{30} - 1 = 1023 \times \textit{something}.$$

It is not prime!

Question: Is any number of the form $x^{30} - 1$ prime? Is $2^{400} - 1$ prime?

Question: Put $x = 10$ into the formula $\left(2x^2 + x + 3\right)\left(2x + 1\right) = 4x^3 + 4x^2 + 7x + 3$ What do you see for ordinary arithmetic?

COMMON CORE MATHEMATICAL PRACTICE STANDARDS:
This MAA AMC problem and its discussion certainly follow a number of practice standards:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

(Use this problem as part of a guided class discussion.)

7. Look for and make use of structure.

BONUS: DIVISIBILITY RULES

Although not part of any standard curriculum many people are aware of a number of divisibility rules. They are fun to think about and are worth mentioning here.

(But do note … These rules become joyless when made dictates to be memorized! One can always nut one's way through a divisibility question using common sense and wits: For example: Is 161 divisible by seven? Answer: Think 140.)

Divisibility by 1: Every number is divisible by one. Not much to say here!

Divisibility by 2, 5, and 10: Every number can be written as a multiple of ten plus a final digit. (For example, 6549 equals $6540+9$). And as every multiple of ten is already a multiple of 2, 5 and 10 we need only check the divisibility of the final digit.

Divisibility by 2: *A number is divisible by two only if its final digit is even, that is, if the final digit is 0,2,4,6 or 8.*

Divisibility by 5: *A number is divisible by five only if its final digit is divisible by five, that is, if the final digit is 0 or 5.*

Divisibility by 10: *A number is divisible by ten only if its final digit is divisible by ten, that is, only if the final digit is 0.*

Divisibility by 9 and 3:
Each power of ten, $1, 10, 100, 1000, \ldots$, is one more than a multiple of nine (for example, $1,000 = 999 + 1$) and so each leaves a remainder of one upon division by nine.
Consequently, a number such as $7,328$, which equals
$7\times1000+3\times100+2\times10+8\times1$,
leaves a remainder of
$7\times1+3\times1+2\times1+8\times1=7+3+2+8$
upon division by nine.

Divisibility by 9: *Upon division by nine, a number leaves the same remainder as the sum of its digits. In particular, if the sum of digits is itself a multiple of nine, then so is the original number.*

Exactly the same reasoning works for the number three throughout the discussion above: *Each power of ten* $1, 10, 100, 1000, \ldots$ *is one more than a multiple of three (for example,* $1,000 = 999 + 1$*) and so…* And so exactly the same divisibility rule applies.

Divisibility by 3: *Upon division by three, a number leaves the same remainder as the sum of its digits. In particular, if the sum of digits is itself a multiple of three, then so is the original number.*

For example, we see that $2,348$ leaves a remainder of $2+3+4+8=17$ upon division by three, which is the same as a remainder of $1+7=8$, which is equivalent to a remainder of 2.

Divisibility by 4 and 8:
Every multiple of one hundred is a multiple of four. So to check for divisibility by four, we need check only the final two digits of a number.

For example, 4576 equals $45\times100+76$. 100 is a multiple of four and 76 can be halved twice ($76\to38\to19$) and so too is a multiple of four. Thus 4576 is divisible by four.

Divisibility by 4: *A number is divisible by four if its final two digits represent a two-digit number that is a multiple of four. (Can be halved twice!)*

Since 1000 is a multiple of eight (it equals $10 \times 10 \times 10 = 2 \times 2 \times 2 \times 5^3 = 8 \times 5^3$) we have:

Divisibility by 8: *A number is divisible by eight if its final three digits represent a three-digit number that is a multiple of eight. (Can be halved thrice!)*

Question**:** Devise a divisibility rule for the number 16. Devise one for 32 as well.

Divisibility by 6, 15, 45, 12, … :
For a number to be divisible by six we need it to simultaneously be a multiple of 2 and a multiple of 3.

Divisibility by 6: *A number is divisible by six if it is even (final digit 0,2,4,6 or 8) and its digits sum to a multiple of three.*

In the same way we see:

Divisibility by 45: *A number is divisible by 45 if its final digit is 0 or 5 and all its digits sum to a multiple of nine.*

Question**:** What are appropriate divisibility rules for 12 and 15 and 72?

QUERY: Divisibility Rule for 11?
Observe that 10 is one less than a multiple of 11. We could say this in an unusual way:

> 10 *leaves a remainder of* -1 *upon division by* 11.

Consequently … $100 = 10 \times 10$ leaves a remainder of $(-1) \times (-1) = 1$ upon division by 11. (This is true, as $100 = 9 \times 11 + 1$).

And $1000 = 10 \times 10 \times 10$ leaves a remainder of $(-1) \times (-1) \times (-1) = -1$ upon division by 11. (And this is true: $1000 = 91 \times 11 + (-1)$.)

And $10000 = 10 \times 10 \times 10 \times 10$ leaves a remainder of $(-1)^4 = -1$.

And so on.

What remainder then does $7,328$ leave upon division by 11?
(Think $7 \times 1000 + 3 \times 100 + 2 \times 10 + 8 \times 1$.)

Devise a general divisibility rule for the number 11.
(See www.jamestanton.com/?p=1287 for a video on this.)

Even More Divisibility Rules!
There is a divisibility rule for 7. There are divisibility rules for 13, 17, 19 and so on! Can you devise such rules for yourself?

[Some rules for these appear at www.jamestanton.com/?p=1287 .]