CURRICULUM INSPIRATIONS

from the MAA AMC

TIDBITS FOR THE CLASSROOM INSPIRED BY MAA AMERICAN MATHEMATICS COMPETITIONS

ESSAY NUMBER: 5

Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty. (From amc.maa.org.)

The MAA AMC has developed, over the past six decades, thousands of fabulous mathematical delights, of all flavors, for all levels of mathematical intrigue. And it is all directly relevant for the classroom! We can teach and support curriculum content AND, more important, we can teach the art of creative and original enquiry. As I recently wrote to some fellow mathematics educators, my greatest wish for each of our next generation of students is:
…a personal sense of curiosity coupled with the confidence to wonder, explore, try, get it wrong, flail, go on tangents, make connections, be flummoxed, try, wait for epiphanies, lay groundwork for epiphanies, go down false leads, find moments of success nonetheless, savor the "ahas," revel in success, and yearn for more.

(Sounds like the right way to go about life in general!) The culture of the typical classroom, however, pressed by the weight of mandated tests and exams, may feel the opposite of this! (*There simply isn't time to play with ideas*). But we can introduce moments of rich, mindful engagement. The MAA AMC materials provide such moments.

A Word on Words: Speaking of mathematics education culture … Take note of the words we regularly use for moments of enquiry in the classroom:

exercise problem challenge
worksheet exam

We have students engage in *competition*. We discuss how we might go about *attacking* a problem. It's all combative and negative! How did this come to be in mathematics?

Rather than "attack a problem" can we perhaps "probe an idea"? Rather than do twenty exercises for homework (and why not add twenty push-ups to that as well?) can we engage in explorations?

Many folk are uncomfortable with mathematics competitions precisely because of their competitive sense. (*Mathematics is hard enough as it is. Why set students up for an even more direct sense of failure?)* Others object to the focus on speed and the focus on answering "what" questions. (*Where's the mulling, the deeper exploration, the reflection on connections to old ideas and the inspiration for new ones?*)

Let go of all that here. The goal of this essay, and all the essays in this series, is to connect with joyful learning and interesting mulling. That is, to connect with the art of mathematics doing. The material of MAA AMC can show how.

Our tidbit today is problem 20 from the 2007 AMC 10 B competition. It is considered a challenge for 10th graders.

A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

Let's begin with our standard first two steps to analyzing an interesting enquiry.

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

A deep breath really does help.

Now …

STEP 2: Understand the question. Understand the different components of the question.

To help with this, simply ask:

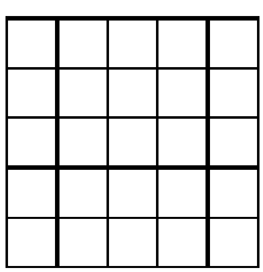
What is the question about? (Just state the obvious!)

It is about a 5×5 grid of square blocks. (Stating the obvious is emotionally safe and gets one over the initial fear of starting.)

This question seems amenable to the problem-solving technique of essay 4:

Can we draw a picture?

Yes. Again, just draw the obvious: a 5×5 grid of square blocks.



What are we doing with this picture?

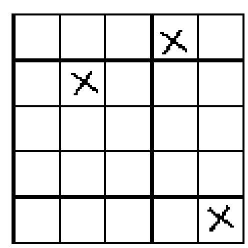
This is the next natural question to ask. To answer it we need to reread the question. (We will no doubt be doing this multiple times!)

We are selecting "a combination" of three blocks, no two of which are in the same column or some row. We want to count how many ways this can be done.

What does this mean? The word "combination" sounds like a math word, but I don't really know what it means. Let's just push on and see if we can just intelligently fumble our way through this.

Aside Comment: So much of the mathematics curriculum places emphasis on mathematics jargon over mathematics doing. We give students the impression that they should know what a "combination" is, for example, and hence set them up for an emotional sense of failure if they do not. I wish we could de-emphasize jargon and focus on developing for students the confidence to rely on their wits, to "nut their way" through ideas, even if they don't know specific words. They can make mighty good educated guesses.

We want three blocks, no two in the same row or same column. It can happen.



And I bet there are lots of ways to do this. The question wants us to count how many ways.

Hmmm. This problem feels too big. It suggests a new problem solving technique.

CAN WE SOLVE A SMALLER VERSION OF THE SAME PROBLEM?

I can think of two smaller versions of the question. (Are there others?)

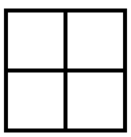
1. Work with a smaller grid of squares.
2. Choose less than three squares in the grid.

Since we don't know where we're going, let's try both!

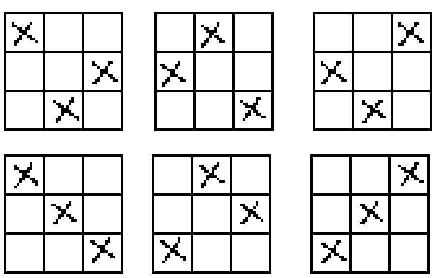
Smaller Problem 1:
How many different ways can we select three blocks, no two in the same row or column ...
... for a 1×1 *grid?*

Okay .. this is silly, but we can certainly solve it. There are zero ways to select three squares in a 1×1 grid.

...for a 2×2 *grid?*
We can certainly select three squares in a 2×2 grid, but some will necessarily lie in the same row and the same column. The answer to the question is zero for a 2×2 grid.



...for a 3×3 *grid?*
Okay, now things are complicated. I don't know how to think about this so I'll just sit down and work them all out.



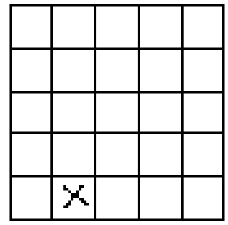
There are six ways to select three squares appropriately on a 3×3 grid.

...for a 4×4 *grid?*
This seems too hard again!

Let's try the other version of a smaller problem.

Smaller Problem 2:
In a 5×5 *grid, how many different ways can we select* ***ONE*** *block, no two in the same row or column?*

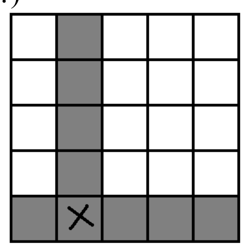
This is easy. There are 25 choices for what the one block could be.



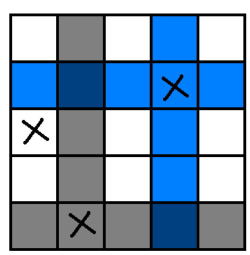
The answer to this small problem is 25.

In a 5×5 *grid, how many different ways can we select* ***TWO*** *blocks, no two in the same row or column?*

Hmm. Well, the picture above shows that there are 25 ways to select the first block. How many choices have we for the second block? It can't be in the same row or column as the first choice. This leaves 16 options for the placement of the second block. (Always 16?)



AHA! And if we go for a **THIRD** block as the original question asks, then we see that once the first and second blocks are chosen, we are left with 9 possibilities for the third block. (Always 9?)



25 choices for a first block, 16 choices for a second block, and 9 choices for a third gives a total of $25 \times 16 \times 9 = 3600$ possibilities in all. It seems we have answered the puzzler!

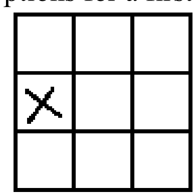
STEP 3: Have we?

In the setting of an exam or a competition, speed is often seen as paramount. Checking one's solution is often relegated to a secondary concern. (And what about engaging in further reflection and exploration?!)

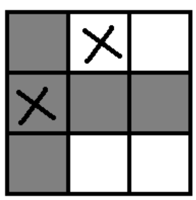
We have an easy means to check our solution here. We selected three blocks from a 3×3 grid and saw, by hand, that there were six possibilities. Does the reasoning we just followed give the same answer?

How many different ways can we select three blocks, no two in the same row or column for a 3×3 *grid?*

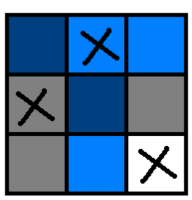
There are 9 options for a first choice.



Once selected, there are 4 options for a second block:



And this leaves just 1 choice for a third block.



According to this reasoning there are thus $9\times 4\times 1=36$ ways to appropriately select 3 blocks in a 3×3 grid.

SOMETHING IS WRONG!

STEP 4: When something is wrong, put the problem aside and go for a walk. Let your subconscious mull on matters for a while.

This is sound advice for a practicing mathematician. Clarity and insight usually comes from not thinking about the conundrum at hand! Alas, the mathematics classroom does not allow for this necessary, genuine practice.

Comment: When my high-school Advanced Topics students were stuck on an idea, I assigned the following homework: *Tonight, go for a twenty-minute walk and don't think about the problem.* I meant this and absolutely expected my students to do it! Class discussions the next day always proved to be mighty fruitful.

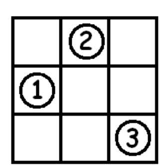
So what is the issue here? In the 3×3 grid we assigned a first block, a second block and a third block and got an answer of 36, which we know is not correct. (The answer is 6, not 36.) I don't trust our answer of 3600 for the 5×5 grid. Hmm.

Mulling some more … For the 3×3 problem we got more a significantly bigger answer than we should have. We must have somehow done more than what the question asked for. What more? We chose a first block, a second block and a third block with no two in the same row or column. That sounds right!

At this point a walk really might help.

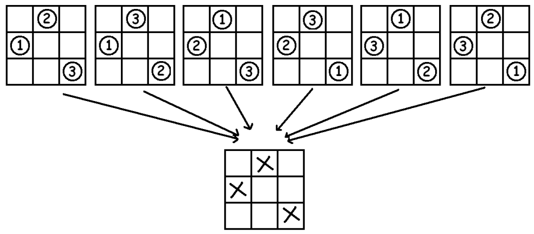
As a teacher guiding a group class-discussion along these lines, a class walk might not be needed. With many minds chances are that at least one student will make a comment that leads to the break-through:

Epiphany: *We actually counted solutions that look like this:*



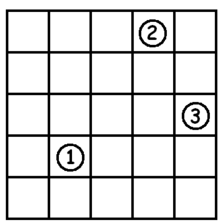
Our X *s were numbered!*

There are six ways the three Xs could be been labeled as "chosen first," "chosen second" and "chosen third" (be clear on this!) and all six numberings correspond to same picture with un-numbered Xs.



We have counted six times as many solutions! So the number of ways to select three blocks from a 3×3 grid, with no two in the same row or column is 36 divided by six, which is 6. Bingo!

For the 5×5 grid exactly the same issue is at hand. We counted "solutions" that look like this:



and so again we are off by a factor of six.

The correct solution to the original problem is: *There are* $3600\div6=600$ *ways to select three blocks from a* 5×5 *grid with no two in same row or same column.*

Phew!

COMMON CORE STANDARDS and PRACTICES:

This problem is about counting. It uses the Fundamental Principle of Counting multiple times:

If there are a *ways to complete a first task and* b *ways to complete a second (and no outcome of the first task in any way influences the choice of outcomes for the second task), then there are* $a\times b$ *ways to complete both tasks together.*

This principle is not explicitly mentioned in the Common Core State Standards (nor are "permutations" and "combinations"), but its study is implied, for example, in the high-school standard:

S-MD-3: *Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated, find the expected value.*

For some help on teaching the Fundamental Principle of Counting, permutations and combinations, see www.jamestanton.com/?p=659 .

The discussion in this essay is right on the mark with the following practice standards:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with Mathematics
7. Look for and make use of structure.

(Drawing a picture is a valid, albeit basic, form of modeling!)

EXTRA FUN!

PUZZLER: *In how many ways can one place eight rooks on a chessboard in mutual non-attack?*
(That is, in now many ways can one place eight counters on an 8×8 grid with no two counters in the same row or the same column?)

FAMOUS PUZZLER: *Is it possible to place eight queens on a chessboard in mutual non-attack?* (And if the answer is "yes," dare I ask in how many ways?)

Care to ask about placing rooks or queens on other sized-boards? Rectangular boards?