Curriculum Inspirations

Inspiring students with rich content from the MAA American Mathematics Competitions

MAA

Curriculum Burst 20: Two-Digit Means

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The arithmetic mean of two distinct positive integers x and y is a two-digit number. The geometric mean of x and y is obtained by reversing the digits of the arithmetic mean. What is $|x-y|$?

SOURCE: This is question # 21 from the 2011 MAA AMC 12b Competition.

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Algebra

COMMON CORE STATE STANDARDS

A-SSE.2: Use the structure of an expression to identify ways to rewrite it.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGIES

ESSAY 1: **ENGAGE IN SUCCESSFUL FLAILING**

THE PROBLEM-SOLVING PROCESS:

Always the first step …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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As I read the question I feel I understand its components.

We are told $\frac{x+y}{2}$, the arithmetic mean, is a two-digit number and $\sqrt{xy}$, the geometric mean, turns out to be the reverse of that number. Weird, but comprehensible.

But I worry that I am meant to find an actual value to the absolute value $|x-y|$ and we don't have a number in sight!

Alright, let's just **DO SOMETHING** and see if anything comes out of our mathematical ramblings.

How could I express that $\frac{x+y}{2}$ is a two-digit number?

Perhaps write : $\frac{x+y}{2} = 10a+b$ where a and b are each single digits. Then the reverse of this two-digit number is:

$$\sqrt{xy} = 10b + a .$$

It feels irresistible to square this:

$$xy = 100b^2 + 20ab + a^2 .$$

Should we square $\frac{x+y}{2} = 10a+b$ as well?

The question calls for the absolute value $|x-y|$ ad absolute values are notoriously difficult to work with. One way to handle them is to square them and make the absolute value considerations go away! So let's try to find the value of

$$|x-y|^2 = (x-y)^2 = x^2 - 2xy + y^2$$

instead. Ooh! We already have a formula for xy.

Okay, let's do square $x + y = 20a + 2b$. It does seem as though it might help.

$$x^2 + 2xy + y^2 = 200a^2 + 80ab + 4b^2$$

Okay, I can piece things together:

$$\begin{aligned} |x-y|^2 &= x^2 + 2xy + y^2 - 4xy \\ &= 400a^2 + 80ab + 4b^2 - 400b^2 - 80ab - 4a^2 \\ &= 396a^2 - 396b^2 \\ &= 396\left(a^2 - b^2\right) \end{aligned}$$

Whoa! With the difference of two squares formula I can write:

$$|x-y|^2 = 396(a+b)(a-b)$$

That felt good, but what does it do for us? I need to get an actual number from this. All I know is that a and b are each single digits: 0, 1, 2, … or 9.

The left side is a perfect square. So too must be the right side. Is 396 a square number? Hmm. $396 = 11 \times 36$. It has a factor of 11.

Okay, so $36 \cdot 11 \cdot (a+b) \cdot (a-b)$ must be a perfect square. That prime number of 11 sitting there means that one of $a+b$ or $a-b$ must also have a factor of 11. Since a and b are each single digits $a-b$ has value between 0 and 9 and $a+b$ has value between 1 and 18. The only extra factor of 11 in $36 \cdot 11 \cdot (a+b) \cdot (a-b)$ must be coming from $a+b$ being 11. The other number, $a-b$, must then be 1, 4 or 9.

$$\begin{aligned} a+b &= 11 \\ a-b &= 1,\ 4,\ or\ 9 \end{aligned}$$

Adding gives $2a = 12,\ 15\ or\ 20$. Only $a = 6$, and hence $b = 5$, works for the problem. It all fell into place!

$$|x-y|^2 = 396(a+b)(a-b) = 36 \cdot 11 \cdot 11 \cdot 1$$

and so $|x-y| = 6 \cdot 11 = 66$. Whoa!

Extension: i) How did the author of this question discover this question? The numbers worked out extraordinarily beautifully! ii) Is there a three-digit version of this problem?

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