Curriculum Burst 16: Quadratic Values

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Let $f(x)=ax^2+bx+c$, where a, b, and c are integers. Suppose that $f(1)=0$, $50<f(7)<60$, $70<f(8)<80$, and $5000k<f(100)<5000(k+1)$ for some integer k. What is k?

SOURCE: This is question # 20 from the 2011 MAA AMC 12a Competition.

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Function Notation; Quadratics and Polynomials

COMMON CORE STATE STANDARDS

F-IF.2: Use function notation, evaluate functions for inputs in their domains and interpret statements that use function notation in terms of a context.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGIES

ESSAY 2: **DO SOMETHING**
ESSAY 3: **ENGAGE IN WISHFUL THINKING**

THE PROBLEM-SOLVING PROCESS:

The right place to begin…

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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I feel like I can "see through" this question. It is about a quadratic $ax^2 + bx + c$ (and I have studied quadratics in great depth in algebra II) with a whole bunch of complicated details that, in the end, seem only to be about plugging in numbers. That feels do-able. So I am just going to cross my fingers and follow my nose on this one and just start with the strategy…

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DO SOMETHING
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Okay, reading through the question now with care, I see we have a quadratic:

$$f(x) = ax^2 + bx + c.$$

And we are first told: $f(1) = 0$. No problem, this means:

$$a + b + c = 0.$$

Next we are told some complicated things about $f(7)$ and $f(8)$. Well …

$$f(7) = 49a + 7b + c$$

$$f(8) = 64a + 8b + c$$

I am not sure what's next. What specifically are we being told about $f(7)$ and $f(8)$?

Now $50 < f(7) < 60$ is telling me that $f(7)$ is a number in the 50s. (Is it obvious that $f(7)$ is an integer?) And $70 < f(8) < 80$ says $f(8)$ is an integer in the 70s.

Let me write:

$$49a + 7b + c = \text{fifty something}$$
$$64a + 8b + c = \text{seventy something}$$

We still have:

$$a + b + c = 0$$

I am not sure where this is taking me. But it does look like a system of three equations in three unknowns (with extra "unknownishness" of where exactly I am in the fifties and the seventies!)

Shall we just try some standard algebra: subtract one equation from another to eliminate a variable? We should make use of the equation with the zero in it.

Subtracting this third equation from the first gives:

$$\begin{array}{rl} 49a + 7b + c = & \text{fifty something} \\ a \quad + b \quad + c = & 0 \\ \hline 48a + 6b \qquad = & \text{fifty something} \end{array}$$

Helpful? Hmm. Subtracting the third equation from the second gives:

$$63a + 7b = \text{seventy something}$$

I am still not sure if this is at all helpful.

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ENGAGE IN WISHFUL THINKING
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We have:

$$48a + 6b = \text{fifty something}$$
$$63a + 7b = \text{seventy something}$$

If we knew what the actual numbers are on the right, we could then solve for a and b and use $c = -a - b$ to find c. Then we would know $f(x)$ completely and we could just compute $f(100)$ to solve the problem! Is there any way to know those numbers?

Oh heavens! $63a + 7b$ is a multiple of seven, and it must be a multiple of seven in the seventies (and not be 70 itself). It can only be 77!

What about $48a + 6b$? It is a multiple of six in the fifties. It can only be 54! (The author of this question was very clever!)

Solving gives $a = 2$, $b = -7$ and $c = 5$. So $f(100) = 2(100)^2 - 7(100) + 5 = 19{,}305$ and which is between the third and fourth multiples of 5000. So $k = 3$. Wow!

Extension: Design an equally clever problem like this, but for a cubic!

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