Curriculum Burst 15: Units Digit

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What is the units digit of $19^{19} + 99^{99}$?

SOURCE: This is question # 14 from the 1999 MAA AMC 8 Competition.

QUICK STATS:

MAA AMC GRADE LEVEL
This question is appropriate for the 8^{th} grade level.

MATHEMATICAL TOPICS
Integer Exponents; Seeing Structure in Expressions

COMMON CORE STATE STANDARDS

6.EE.1: Write and evaluate numerical expressions involving whole-number exponents.

A-SSE.1b: Interpret complicated expressions by viewing one or more of their parts as a single entity.

MATHEMATICAL PRACTICE STANDARDS
MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.

PROBLEM SOLVING STRATEGIES
ESSAY 2: **DO SOMETHING**
ESSAY 5: **SOLVE A SMALLER VERSION OF THE SAME PROBLEM**
ESSAY 8: **AVOID HARD WORK**

THE PROBLEM-SOLVING PROCESS:

The most important step …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

There is no way I want to work out 19^{19} and 99^{99} ! There has to be an easier way than doing all the arithmetic. But I don't see what that easier way could be.

REREAD THE QUESTION

The question is only asking for the final digit of $19^{19} + 99^{99}$
Would that be $9^{19} + 9^{99}$? I don't know. (I am just guessing.) But even if that is right I still don't want to work out those numbers!

TRY A SMALLER VERSION OF THE SAME PROBLEM.

What if I made the exponents smaller – just to get a feel for the problem? Say, looked at $19^3 + 99^3$? Still too hard. How about just $19 + 99$? The final digit there comes from adding the nines.

$$\begin{array}{r} {}^{1}\\ 19 \\ +\ 99 \\ \hline \text{etc } 8 \end{array}$$

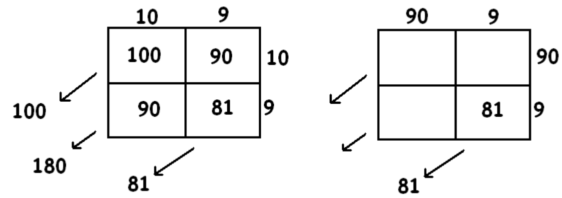
(I don't need to do the arithmetic all the way through to see this.) $19 + 99$ has final digit 8 .

Actually I never have to do the arithmetic of a sum all the way through to see what the final digit is going to be. I need only add the final digits of the numbers.

Okay then, how about $19^2 + 99^2$? I could work these out.

AVOID HARD WORK.

I need only compute 19^2 and 99^2 far enough to see their final digits.



I can see that 19^2 and 99^2 each end in a 1. Thus $19^2 + 99^2$ ends in a 2 .

Okay, how about $19^3 + 99^3$? Well…
$19^3 = (10+9)^3 = (10+9)(10+9)(10+9)$ If I expand this out, any product involving one or more of the 10s will "skip" the final digit. The only product that lets me see it is $9 \times 9 \times 9$.

Now $9 \times 9 \times 9 = 81 \times 9$ and so ends with a 9 . This means 19^3 also ends with a 9 . And so too does $99^3 = (90+9)(90+9)(90+9)$ by exactly the same line of reasoning! Wow! $19^3 + 99^3$ is a sum of two numbers each ending with 9 . And so it ends with 8 .

I guess we need to check $19^4 = (10+9)^4$ and $99^4 = (90+9)^4$ now. But writing out the products in full we will see that only the product $9^4 = 9 \times 9 \times 9 \times 9 = 81 \times 81$, which ends with a 1, reveals the final digits. We have $19^4 + 99^4$ ends with $1 + 1 = 2$.

Since 9^4 ends with 1, $9^5 = 9^4 \times 9$ ends with 9 . Since 9^5 ends with 9 , $9^6 = 9^5 \times 9$ ends with 1. Since 9^6 ends with 1, $9^7 = 9^6 \times 9$ ends with 9 . And so on.

We have: 9^{odd} ends with 9 and 9^{even} ends with 1, and the same pattern holds for powers of 19 and of 99 .

So 19^{19} ends with 9 , 99^{99} ends with 9 . Thus $19^{19} + 99^{99}$ ends with 8 !

Extension:
What is the first digit of $19^{19} + 99^{99}$?

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