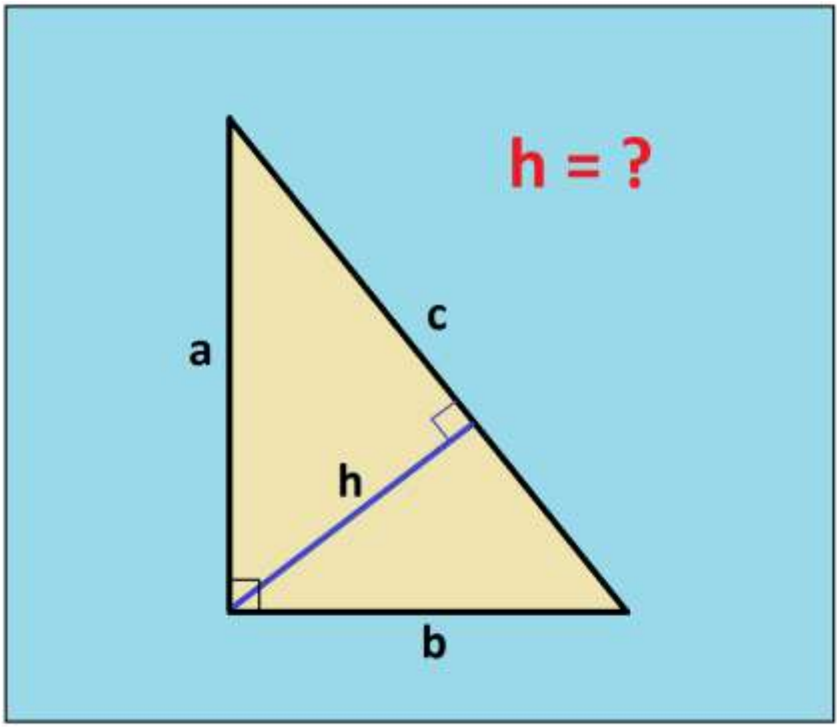


TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

BY JAMES TANTON

WWW.MAA.ORG/CI

A Distance Formula

With its focus on similarity, the standard geometry curriculum often sidesteps the power of area thinking for computing lengths in triangles. I nice example of this is the following question.

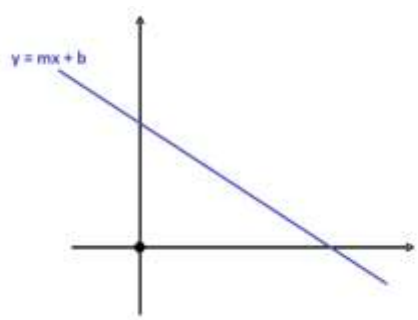
Exercise: *A right triangle has legs of lengths* a *and* b *and hypotenuse* c. *Find a formula, in terms of* a, b, *and* c, *for the length* h *of the non-trivial altitude of the triangle.*

Answer: Computing the area of the triangle two different ways gives $\frac{1}{2}ab = \frac{1}{2}hc$, from which we see $h = \frac{ab}{c}$.

(Here's an interesting question: *Is there an integer right triangle with* h *also of integer value?*)

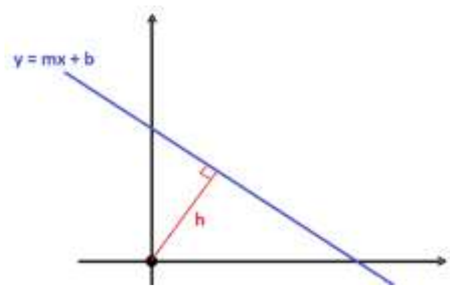
One can use this question to promote clever thinking in analytic geometry.

Exercise: *A line* $y = mx + b$ *has positive* x *- and* y *-intercepts. Find a formula, in terms of* m *and* b, *for the distance of the origin from the line.*



Putting these two examples side-by-side actually illustrates a good problem-solving strategy. **SECOND-GUESS THE AUTHOR!** Let students wonder why these two questions are together.

This second question seeks the length of the altitude of a right triangle. The legs of the triangle have lengths b and $\frac{b}{-m}$ (the slope m is negative), and hypotenuse $\sqrt{b^2 + \frac{b^2}{m^2}}$.



So the distance h of the origin from the line is

$$h = \frac{b \times \frac{b}{-m}}{\sqrt{b^2 + \frac{b^2}{m^2}}} = \frac{b^2}{\sqrt{b^2 m^2 + b^2}} = \frac{b}{\sqrt{m^2 + 1}}.$$

(Does this feel right? The formula shows that if we double the value of b, the distance of the line from the origin also doubles. Does that seem plausible?)

A natural question: *How does this formula change for lines that make right triangles in different quadrants of the plane?* That is, what if the sign of b can be either positive or negative, as can be the sign of m?

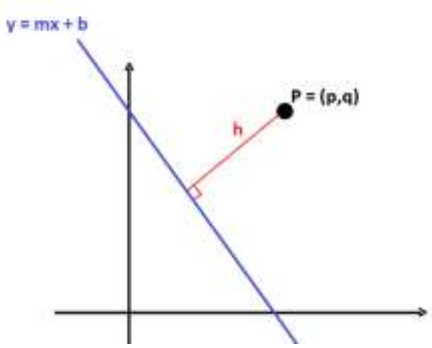
One checks that in each case, one has a right triangle with legs of lengths $|b|$ and $|\frac{b}{m}|$ and hypotenuse $\sqrt{b^2+\frac{b^2}{m^2}}$. The same algebra then gives

$$h=\frac{|b|}{\sqrt{m^2+1}}.$$

This formula is correct even if the line passes through the origin. (In this case $b=0$ and the distance of the origin from the line is 0.)

(**Exercise:** With a geometry software package, plot a large sample of lines of the form $y=mx\pm\sqrt{m^2+1}$ for different positive and negative values of m. Can you predict what you should see?)

We've found the formula for the distance of the origin from a given line. What about a formula for the distance h of a general point $P=(p,q)$ from a line?



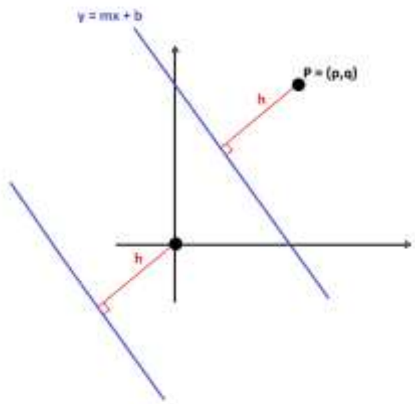
This now seems hard!

ENGAGE IN WISHFUL THINKING

The fact that $P=(p,q)$ is not the origin makes this problem difficult. We would have the problem licked if it were.

Here's a crazy idea: Could just subtract p from all the x-coordinates of points and q from all y-coordinates of points? If so, then $P=(p,q)$ would become $(0,0)$ and we'd be working with the origin! Can we do such a thing?

Well, changing all the x- and y-coordinates of points this way is tantamount to performing a geometric translation. And indeed, this will take P to the origin and no distances will change as a result!



Now the question is: What is the equation of the translated line?

PERSEVERANCE IS KEY

Well, the translated line has the same slope m and so will be of the form $y=mx+B$ for some value B. Can we work out that value? Do we know the y-intercept of the translated line?

Hmm. That seems hard.

Do we know the coordinates of any point that the translated line goes through?

Actually … yes! The original line passes through the point $(0,b)$ and so the translated line passes through the translated point $(-p,b-q)$. This gives

$$b-q=m(-p)+B$$

yielding

$$B=mp+b-q.$$

So the equation of the translated line is

$$y=mx+(mp+b-q).$$

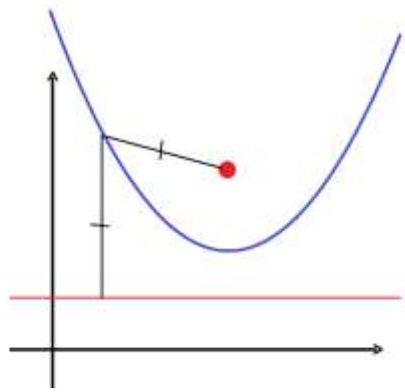
We're now set! The distance of the origin from this line is

$$h=\frac{|mp+b-q|}{\sqrt{m^2+1}}$$

and so the distance of $P=(p,q)$ from the line $y=mx+b$ is given by this formula too!

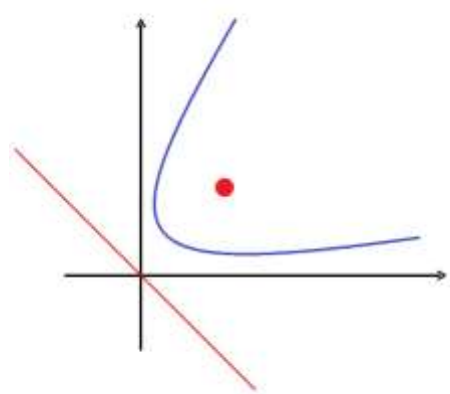
TILTED PARABOLAS

A *parabola* is defined as the locus of points equidistant from a given line (called its *directrix*) and a given point (called its *focus*). In an advanced algebra class one learns that the graph of a quadratic equation $y=ax^2+bx+c$ turns out to be a parabola with some fixed focus and some horizontal line as its directrix. (Finding the equations for these is not fun!)



The distance of a point (x,y) from a fixed point (a,b) is $\sqrt{(x-a)^2+(y-b)^2}$ and we have just worked out the distance of a point from a fixed line. We are set to try things like the following.

Exercise: *Write down an equation for a* 45° *tilted parabola with directrix the diagonal line* $y=-x$ *and focus* $(1,1)$.



Exercise: *Is the general equation of a parabola with a vertical directrix indeed an equation of the form* $x=ay^2+by+c$*?*