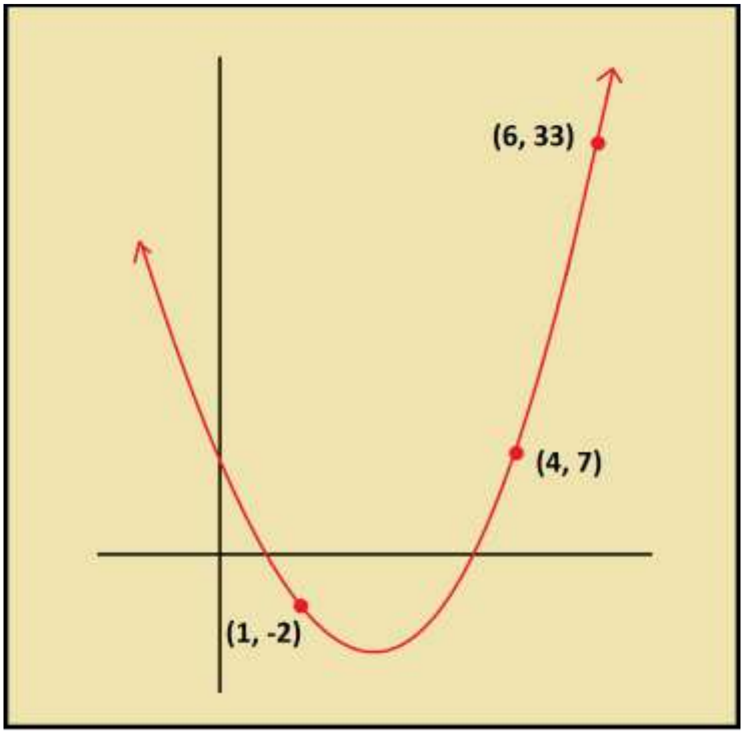

TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

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Fitting Quadratic Functions

The high-school algebra curriculum might ask students to find a quadratic equation that fits three given data points.

Example: *Find a quadratic equation that fits this given data.*

x	y
1	-2
4	7
6	33

The standard school approach is to set up a system of three equations in three unknowns. In this example, if $y = ax^2 + bx + c$ fits the data, then

$$\begin{aligned} a+b+c &= -2 \\ 16a+4b+c &= 7 \\ 36a+6b+c &= 33. \end{aligned}$$

To solve the system, students are typically taught to enter lists and matrices into a scientific calculator, perform a black box operation of "inverting the matrix," and then to read off the answer from a list. One might question the pedagogical value of this calculator work.

Here's another approach. It is deliberately mysterious, designed to help students practice looking for structure in equations and develop the patience to unravel a mystery. It also illustrates the problem-solving strategy: **WORK BACKWARDS**. The approach is simply to make sense of a given answer to the problem.

And here it is. Here's a quadratic equation that fits the data in our example.

$$y=-2\cdot\frac{(x-4)(x-6)}{(-3)(-5)}+7\cdot\frac{(x-1)(x-6)}{(3)(-2)}+33\cdot\frac{(x-1)(x-4)}{(5)(2)}$$

Whoa!

One's first reaction to seeing an answer like this might well be dismay. It looks visually overwhelming.

Deep breath.

After a (valid) emotional reaction we do come to notice that there is structure to the equation: the right side is the sum of three terms with coefficients -2, 7, and 33, for example.

The fraction terms with each coefficient are, however, confusing.

PROBING THE ANSWER

Is the equation actually quadratic?

Perhaps conducting a tiny bit of arithmetic within the equation might be helpful. (**DO SOMETHING!**) We get

$$y=-\frac{2}{15}(x-4)(x-6)-\frac{7}{6}(x-1)(x-6)+\frac{33}{10}(x-1)(x-4).$$

It is now clear that in expanding each of the three terms and summing will indeed give an equation of the form $y = ax^2 + bx + c$.

Is the equation doing what we want it to do?

Does putting in $x = 1$ yield $y = -2$, putting in $x = 4$ yield $4 = 7$, and putting in $x = 6$ yield $y = 33$?

The natural response to this query is … try it!

If we put $x = 1$ into the original equation we see that second and third terms vanish (the expression $x-1$ in each numerator ensures this) and so we are left with

$$y=-2\cdot\frac{(1-4)(1-6)}{(-3)(-5)}+0+0\cdot$$

Notice that the denominator in this first term matches the numerator for $x=1$, and so we have

$$y=-2\cdot 1+0+0=-2,$$

as hoped.

Now let's try putting in $x=4$. Again, we see that two of the three terms in the equation vanishes to give

$$y=0+7\cdot\frac{(4-1)(4-6)}{(3)(-2)}+0=7.$$

And putting in $x=6$ gives

$$y=0+0+33\cdot\frac{(6-1)(6-4)}{(5)(2)}=33.$$

We see now how the original expression was constructed! We wrote y as a sum of terms with each term designed to vanish at all but one given input x-value and to give the desired matching output times a fraction equal to 1 for that input.

For example, a term that vanishes at $x=63$ and $x=102$ but gives the value 5562 at $x=887$ is

$$5562\cdot\frac{(x-63)(x-102)}{(887-63)(887-102)}.$$

Example: *Write down a quadratic function that has the values* A, B, C *at* $x=a$, $x=b$, *and* $x=c$, *respectively.*

Answer:

$$y=A\frac{(x-b)(x-c)}{(a-b)(a-c)}+B\frac{(x-a)(x-c)}{(b-a)(b-c)}+C\frac{(x-a)(x-b)}{(c-a)(c-b)}$$

does the trick.

Making these expressions look friendlier is usually not as tedious as one first fears.

Example: *Make*

$$y=-2\cdot\frac{(x-4)(x-6)}{(-3)(-5)}+7\cdot\frac{(x-1)(x-6)}{(3)(-2)}+33\cdot\frac{(x-1)(x-4)}{(5)(2)}$$

look more manageable.

Answer: We saw that a first round of basic arithmetic helps.

$$y=-\frac{2}{15}(x-4)(x-6)-\frac{7}{6}(x-1)(x-6)+\frac{33}{10}(x-1)(x-4)$$

Now, in expanding, we see that that the coefficient of the x^2 term will be

$$-\frac{2}{15}-\frac{7}{6}+\frac{33}{10}=\frac{-4-35+99}{30}=2.$$

The constant term will be

$$\frac{-2\times 24}{15}-\frac{7\times 6}{6}+\frac{33\times 4}{10}=3.$$

We can work out the coefficient of the x this way too, but the arithmetic of fractions is starting to feel daunting. Let's **AVOID HARD WORK.**

Right now we have $y=2x^2+bx+3$ for some value b. Putting in $x=1$, $y=-2$ gives $b=-7$. Done! We have $y=2x^2-7x+3$.

MORE TO PROBE

This use of the *Lagrange's Interpolation Formula* begs questions.

Exercise: *Suppose three data points lie on a line, as for*

x	2	5	8
y	3	6	9

for instance. Does the Lagrange Interpolation Formula still give a quadratic equation? Try it!

Exercise: *There is no function that matches this data set. In what way does Lagrange's Interpolation Formula break down?*

x	5	7	5
y	1	8	19

Can we instead find an equation of the form $x=ay^2+by+c$ *that fits this data?*

Exercise: *Surely there is nothing special here about quadratics for three data points. Can we find a cubic polynomial that fits this data set?*

x	1	3	4	8
y	9	2	4	5

Can we write down the equation of a line (a degree 1 *polynomial) that passes through the points* $(56,87)$ *and* $(909,7)$*?*

Exercise: *Is the quadratic equation we found that fits the data*

x	y
1	-2
4	7
6	33

unique? Could there be a second quadratic equation that also works?

In general, for any $n+1$ *points in the plane, no two of which lying on the same vertical line, Lagrange's Interpolation Formula shows that there is a degree* n *polynomial whose graph passes through each of them. Is this polynomial unique?*

We can design interesting mini-explorations. For example, can we find a polynomial that "wants to be" $P(x)=\frac{1}{x}$?

Exercise: *A degree* $n-1$ *polynomial* P *has* $P(1)=1$, $P(2)=\frac{1}{2}$, $P(3)=\frac{1}{3}$, …, $P(n)=\frac{1}{n}$ *for positive integer* n. *What is the value of* $P(n+1)$*?*