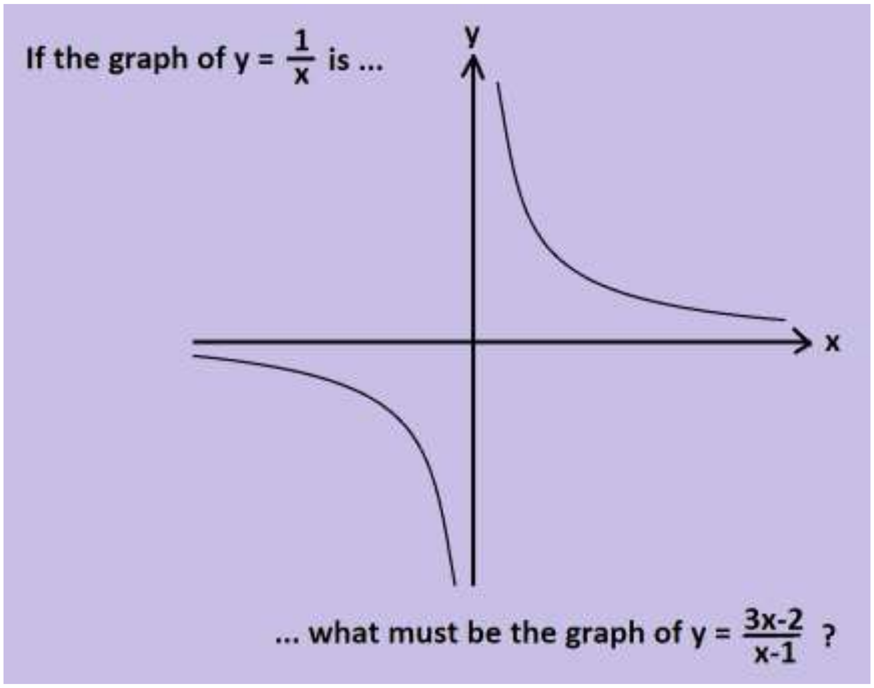

TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

MATHEMATICAL ASSOCIATION OF AMERICA

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Graphing Rational Functions

The upper algebra curriculum has students sketch the graphs of rational functions. The sequence of steps we ask students to follow is really quite long.

Example: Sketch a graph of $y = \dfrac{3x-2}{x-1}$.

Answer:

1) x-intercepts.
We have $y = 0$ when $x = 2/3$.

2) y-intercepts.
We have $y = 2$ when $x = 0$.

3) Long-term behavior.

Rewrite $y = \dfrac{3x-2}{x-1}$ as

$$y = \frac{x\left(3-\frac{2}{x}\right)}{x\left(1-\frac{1}{x}\right)} = \frac{3-\frac{2}{x}}{1-\frac{1}{x}}$$

for values of x different from zero.

We see that if x is large and positive, then the matching y value is close to $\dfrac{3-0}{1-0} = 3$. Thus we have a horizontal line asymptote to the right, the line $y = 3$.

We see the same is true if x is large and negative, and so we have the same horizontal asymptote to the left as well.

4) Vertical Asymptotes.

The denominator in the rational expression is zero for $x = 1$ (and the numerator is not). Thus we have a vertical asymptote, the line $x = 1$.

Just to the left of $x = 1$, say at $x = 0.9979$, we have

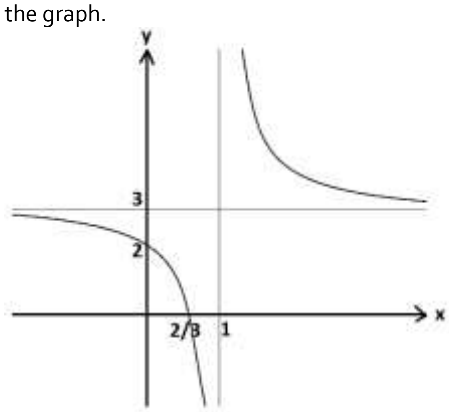
$$y = \text{"}\frac{\text{positive}}{\text{small negative}}\text{"} = \text{"large negative"}$$

and just to the right of $x = 1$, say at $x = 1.002$, we have

$$y = \text{"}\frac{\text{positive}}{\text{small positive}}\text{"} = \text{"large positive."}$$

5) Sketch.

We have enough information for a sense of the graph.

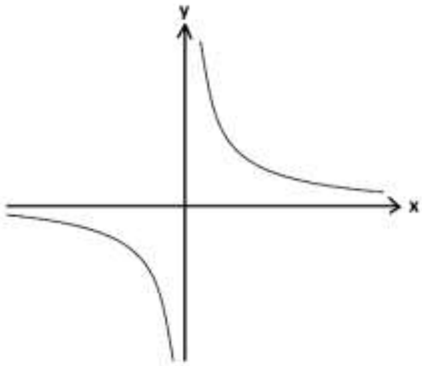


After such an example, ask students to sketch a sampler of such graphs, say, of $y = \dfrac{5x+16}{x+3}$ and $y = \dfrac{1}{x}$, and more. A perfect opportunity then arises inviting students to practice the art of being a mathematician:

AVOID HARD WORK!

Mathematicians will work very hard to figure out ways to avoid hard work.

One notices that all these graphs have the same basic shape, the shape of the graph of $y = 1/x$.



Why must this be so?

Look at $y = \dfrac{3x-2}{x-1}$. Rewrite the rational expression so that a multiple of the denominator appears in the numerator.

$$y = \frac{3(x-1)+3-2}{x-1}$$
$$= 3 + \frac{1}{x-1}$$

We now see the equation as a transformation of the equation $y = \dfrac{1}{x}$, with $x = 1$ "behaving like zero" and all the y-values increased by three. So if we translate the graph of $y = 1/x$ one unit horizontally and three units vertically, we must obtain the graph of $y = (3x-2)/(x-1)$. (And we do! Look at the graph we obtained.)

Similarly, the graph of

$$y = \frac{5x+16}{x+3} = \frac{5(x+3)+1}{x+3}$$
$$= \frac{1}{x+3} + 5$$

is a translation of the graph of $y = 1/x$.

Challenge:

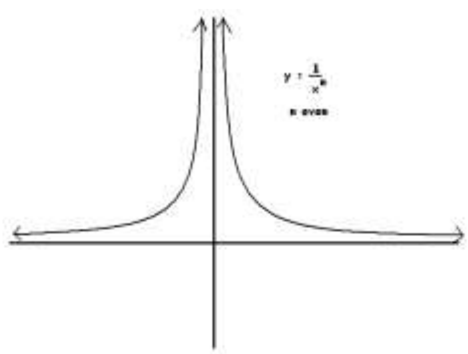
Examine $y = \dfrac{3x-2}{x}$ and $y = \dfrac{x-6}{2x+1}$.

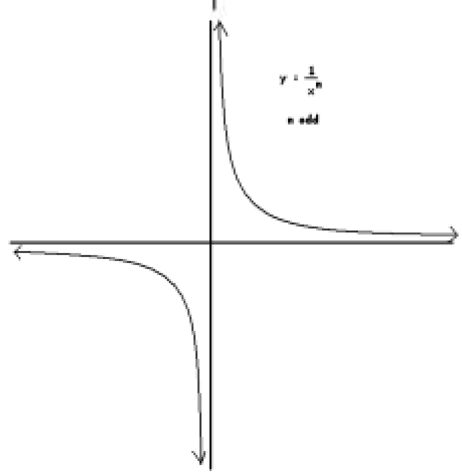
Once we are convinced that every graph of an equation of the form $y = \dfrac{ax+b}{cx+d}$ is a translation (with perhaps a reflection and vertical stretch) of the graph of $y = \dfrac{1}{x}$, one might be able see from inspection what the horizontal asymptote of the graph and the vertical asymptote of the graph must be. (They are, respectively, the lines $y = a/c$ and $x = -d/c$.) Sketching the graph then only takes a few seconds.

Caveat: Speed is not the goal of mathematics! The aim here is to enjoy the process of probing deeply into a structure, namely, the structure of these rational functions, and to see how ideas and insights hang together.

Pushing further: If we know the shape of some basic rational functions, then we can extend this thinking to more general examples.

The graphs of $y = \dfrac{1}{x^n}$ come in two basic shapes, depending on the parity of n.





Graphing $y = \dfrac{2+x^2}{x^2}$, for example, is now straightforward: it is just a translation of the graph of $y = \dfrac{1}{x^2}$.

The graph of $y = \dfrac{3x^2-12x+10}{(x-2)^2}$ is just a transformation of the graph of $y = \dfrac{1}{x^2}$ too. (Do you see how ?)

OTHER TYPES OF ASYMPTOTES

Consider the equation $y = \dfrac{x^3+1}{x}$.

We can rewrite this as $y = x^2 + \dfrac{1}{x}$, as long as we are away from $x = 0$. This reformulation shows that the graph of this equation approaches the graph of the parabola $y = x^2$ for extreme values of x.

(Moreover, $y = x^2 + \dfrac{1}{x} > x^2$ for positive values of x, and $y = x^2 + \dfrac{1}{x} < x^2$ for negative values.)

We still have an asymptote at $x = 0$. We can see that $\dfrac{x^3+1}{x^2}$ is large and positive just to the right of $x = 0$ and large and negative just to the left.

We also deduce that the graph of $y = \dfrac{x^3+1}{x}$ crosses the x-axis at $x = -1$ (and nowhere else).

This feels like plenty of information to garner a sense of its graph.