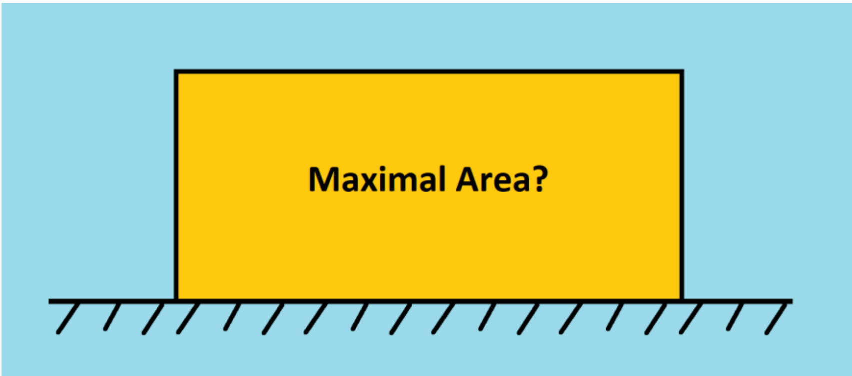

TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

MATHEMATICAL ASSOCIATION OF AMERICA

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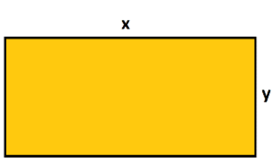
Reflecting on Solutions

Here's a classic question from calculus texts and algebra texts:

A farmer has 40 feet of fencing with which to make a rectangular pen. What are the dimensions of the pen she can make of maximal area?

And here's the classic solution:

Draw a rectangular pen and labels its dimensions x *and* y. *(Always those letters?)*



We can assume the farmer uses all the fencing available (the area of a rectangle can always be increased is leftover fencing is available) so we can assume

$$2x + 2y = 40.$$

This can be rewritten $x + y = 20$.

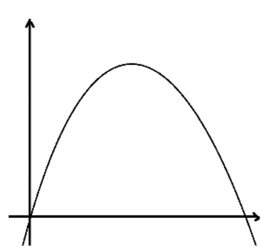
We wish to maximize the area A *of the pen, which is given by*

$$A = xy.$$

Solving for y *in the first equation shows that we want to maximize*

$$A = x(20 - x) = 20x - x^2.$$

Calculus students might differentiate this formula to find the location of the maximum or follow the route of algebra students by recognizing this a quadratic expression whose graph is an inverted parabola.



The maximum occurs at

$$x = \text{"}-\frac{b}{2a}\text{"} = -\frac{20}{-2} = 10.$$

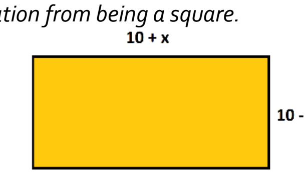
Here $y = 10$ *as well and the pen of maximal area is the* 10*-by-*10 *square.*

Reflection 1: *Might we have expected the symmetrical solution?*

If the pen of maximal area were a non-square rectangle, then we'd expect our mathematics to yield two solutions: one rectangle of maximal area and a second rectangle, its 90-degree rotation, which is a rectangle of the same area. It seems unlikely this problem would have two separate solutions, so maybe we could have guessed that the solution was going to be one invariant under rotation, that is, square? In fact, the picture of the parabola clearly only has one maximum, so we could have deduced right there that the solution had to be symmetrical.

Let's solve the problem again, but this time with matters set up to incorporate the solution we have in mind.

Label the dimensions of a rectangle by its deviation from being a square.



The area of this rectangle is

$$A = (10 - x)(10 + x) = 100 - x^2$$

which is clearly maximal when $x = 0$, *as suspected!*

Swift and lovely!

Back a step:

Reflection 2: *What is the overall structure of what we have done here?*

We started with two variables with a given sum and were asked to maximize their product.

Given:	$P + Q =$ sum
Maximize:	PQ

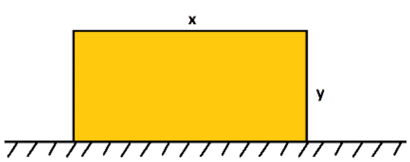
We found that the maximum occurs when we have symmetry, that is, when $P = Q$.

Can we apply this observation to other calculus and algebra problems?

Why yes!

Consider this classic variation of the opening problem.

A farmer has 40 feet of fencing with which to make a rectangular pen. This time the pen will be up against the barn wall. What are the dimensions of the pen she can make of maximal area?



This time, for the labeling shown, we are given

$$x + 2y = 40$$

and we wish to maximize $A = xy$.

But now the equations are not cooperating: the given involves x and $2y$, but the product involves x and y. Bother!

ENGAGE IN WISHFUL THIINKING

I do wish there was a match in the variables for each equation. Wouldn't it be lovely if the product was instead the product $x \cdot (2y)$?

Ahh! We have $x \cdot (2y) = 2xy = 2A$, and maximizing double the area is equivalent to maximizing the area. So let's work instead with this system:

Given:	$x + 2y = 40$
Maximize:	$x \cdot (2y)$

with $P = x$, $Q = 2y$. The maximum occurs when $P = 20$ and $Q = 20$, and so $x = 20$ and $y = 10$ does the trick.

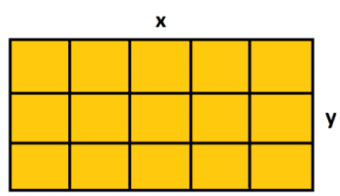
The pen of maximal area is half a square!

Reflection 3: *Could we have suspected the pen here of maximal area will be half a square? What if the barn wall were mirrored?* (Reflect on the role of reflections!)

Reflection 4: Can we go further still?

Consider this wild variation:

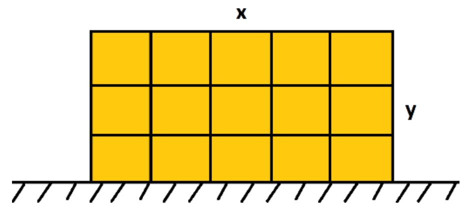
A farmer has 40 feet of fencing with which to make a rectangular pen divided into 15 smaller congruent rectangular pens as shown. What are the dimensions of the pen she can make of maximal total area?



Here we have $4x + 6y = 40$ and we wish to maximize $A = xy$. We might as well maximize $24A = (4x)(6y)$ for which we know the maximum occurs when $4x = 20$ and $6y = 20$, that is, when $x = 5$ feet and $y = 3\frac{1}{3}$ feet.

(These are very small pens!)

Your Turn: Solve the same problem, but this time with the pens built up against a barn wall.



Reflection 5: *In all these problems, is the rectangle of maximal area always the one where the total amount of horizontal fencing used matches the total amount of vertical fencing used? Is that the über message here?*

Reflection 6: *Can we switch things around? Can we say anything about problems that have a fixed rectangle area but now want to optimize the total amount of horizontal and vertical fencing used?*

Perhaps the pedagogical lesson here is that every question solved is an invitation for reflection and further mulling and play. Of course no farmer thinks in the ways suggested by these problems when designing pens. But that is not the point. The content here is not so much the goal of the day, but the art of mathematical thinking invoked is. Let's make sure our mathematics classes, with textbook problems like this, do then attend to that art.