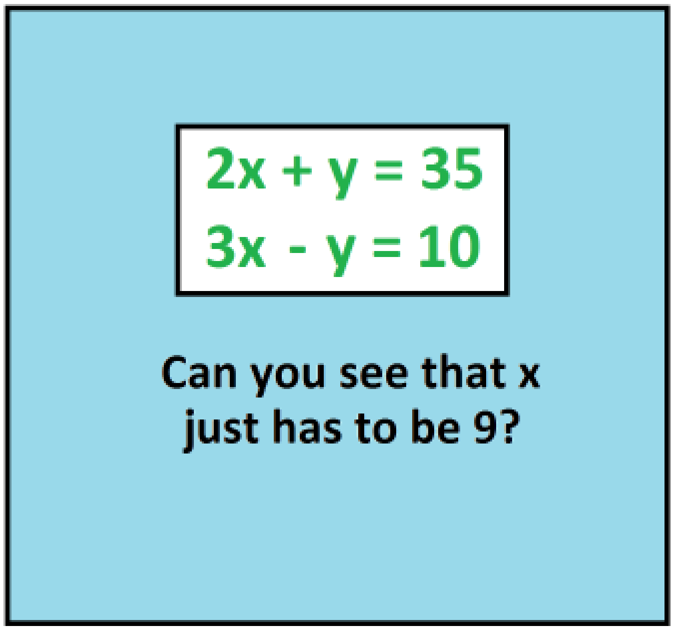

TEACHING THE PROBLEM-SOLVING MINDSET

A Classroom Moment

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Systems of Linear Equations

Many Algebra II curricula have a unit on solving systems of linear equations via algebraic methods. One must, of course, first develop motivation and context for this work (and a good curriculum will subtly establish a need and a desire for wanting to solve systems of equations). But once this is in place, there is still opportunity to re-affirm the problem-solving mindset even when discussing the pure mechanics of the algebra. It need not be rote or algorithmic.

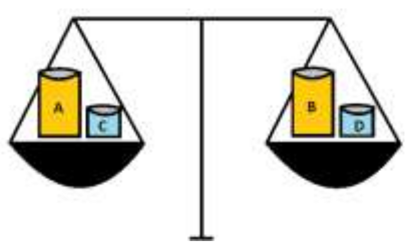
For example, look at the two green equations at the top of this page. Can you see that any pair of numerical values for x and y that makes both number sentences true must have x equal to 9? (And it then follows that y must have value 17.)

It is an epiphany for many students to see that adding the two equations together yields something enlightening. We get $5x = 45$.

QUESTIONING MOMENT:
Can we simply add two equations? If $A = B$ and $C = D$ are we right to conclude that $A + C = B + D$?

$$\begin{array}{lrcl} & A & = & B \\ & C & = & D \\ \hline \text{add} & A+C & = & B+D \quad ? \end{array}$$

This is indeed a property of equality we like to believe. It is often motivated in grade school with a model of scales: If bags A and B weigh the same, and bags C and D do too, then we'll see that bags A and C together match the weight of bags B and D together.



(Others might stress that this property is a consequence of "the property of substitution" in equality.)

With this epiphany at hand, students are now primed to solve a whole slew of simultaneous equation pairs.

Solve: $\begin{array}{l} 2x + 7y = 10 \\ -2x + y = 14 \end{array}$.

Solve: $\begin{array}{l} 5x - y = 9 \\ y - 3x = 8 \end{array}$.

Solve: $\begin{array}{l} 2 + w = 3q - 5 \\ 7 - q = 101w + 2q \end{array}$.

Example: *For which value(s) of* m *does the system*

$$\begin{array}{l} 2x - 7y = 3 \\ mx = 5 - 7y \end{array}$$

fail to have a simultaneous solution?

Notice the little "hiccups" introduced in these problems. They were designed to reinforce the idea that we, as mathematicians, have the power to take control of a given challenge and transmute its details into any alternative form of our liking.

A PROBLEM:

But let's now introduce a real problem.

Solve: *Consider* $\begin{aligned} 4x - \ \ y &= 13 \\ 3x + 2y &= 29 \end{aligned}$.

The approach we developed thus far brought us good success. But it seems to be failing us now. Can our good approach be salvaged in some way?

We now have an opportunity to

ENGAGE IN WISHFUL THINKING.

Wouldn't it be lovely if the top equation possessed the term $-2y$ rather than $-y$?

Well, can we make that happen? Could we make $-2y$ appear in that first equation?

We sure can! Let's multiply the first equation through by two. The system of equations then reads

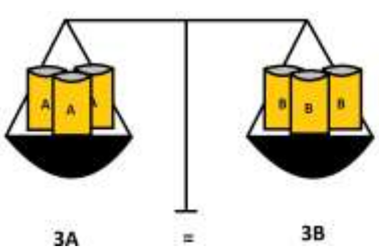
$$\begin{aligned} 8x - 2y &= 26 \\ 3x + 2y &= 29 \end{aligned}$$

from which we see $11x = 55$, giving $x = 5$, and then $y = 7$. Bingo!

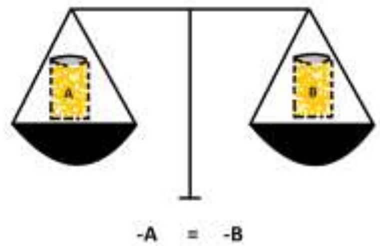
QUESTIONING MOMENT:
Can we simply multiply an equation through by a constant? If $A = B$, does it follow that $kA = kB$ for any given constant k?

This is indeed a second property of equality we like to believe is valid.

It too is often motivated by a scales model in grade school: If bags A and B balance on a scale, then so too will two A bags and two B bags, or twenty-two A bags and twenty-two B bags, or half an A bag and half a B bag; and so on.



And if we fill the bags with anti-matter instead, then a negative A bag will still balance with a negative B bag.



Question: Is $k = 0$ permissible in these considerations?

Solve: $\begin{aligned} 3a - 5b &= 28 \\ 5a + 7b &= 26 \end{aligned}$.

Which turns out to be easier: working to "eliminate a," or working to "eliminate b," or do both approaches require about the same amount of work? (And both approaches do yield the same solution in the end, right?)

Equipped with the power of wishful thinking, students are now all set to solve the standard textbook questions on this topic.

They are also equipped to discover for themselves university Gaussian Elimination if they are up for exploring serious proficiency challenges.

Optional Challenge:
Solve

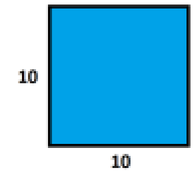
$$\begin{aligned} 2x - y + 3z &= -10 \\ 6x + 5y - z &= 4 \\ -2x + y + 2z &= 5 \end{aligned}$$

Optional Challenge:
Solve

$$\begin{aligned} a + 2b + c + 4d &= 14 \\ 2a + 5b + 3c + 11d &= 25 \\ a + 3b + 3c + 6d &= 16 \\ 3a + 4b - 9c + 19d &= -17 \end{aligned}$$

GOING QUIRKY:

A square of side length 10 has area $A = 100$ square units and circumference (am I required to say *perimeter*?) $C = 40$ units.



There are two famous formulas about area and circumference.

1. Solve the pair of simultaneous equations

$$\begin{aligned} 100 &= \pi r^2 \\ 40 &= 2\pi r \end{aligned}$$

for the unknowns π and r. (Does adding the equations help? How about multiplying or dividing the equations instead?)

What is the value of π for this square? Is there a geometric meaning to the value you get for r?

2. Do all squares have the same π-value? To answer this consider a square with side length s and solve the simultaneous equations

$$\begin{aligned} s^2 &= \pi r^2 \\ 4s &= 2\pi r. \end{aligned}$$

Is a geometric meaning for r now evident?

SPOILER ALERT: We get that the value of π is four for all squares, and that r is the radius of the incircle of the square (the length of the apothem). With these interpretations for π and r we have just shown that the classic formulas $A = \pi r^2$ and $C = 2\pi r$ pertain to squares too!

3. What is the value of π for an equilateral triangle?

4. What is the value of π for a regular N-gon?

SPOILER ALERT: One gets that $\pi = N \tan\left(180^\circ / N\right)$ for a regular N-gon. Put in $N = 1000000$ and you get $\pi \approx 3.141592653...$, a value very close to the value of π for a circle.