TANTON'S TAKE ON …

TEST AND EXAM QUESTIONS

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

NOVEMBER 2012

I was recently asked what my wish was for the next generation of students coming through our mathematics education system. I wrote: *[For each student] a personal sense of curiosity coupled with the confidence to wonder, explore, try, get it wrong, flail, go on tangents, make connections, be flummoxed, try, wait for epiphanies, lay groundwork for epiphanies, go down false leads, find moments of success nonetheless, savor the "ahas," revel in success, and yearn for more.* This might not be the answer one would expect, but it came from the heart without my thinking about it.

So much of the mathematics curriculum in past decades has been focused on skill and "what" questions: *What is measure of* $\angle ABD$*? What is the percentage increase? What is* 357×892*? What is the equation of the tangent line?* And as an educator for the past 11 years I have been working to find the wiggle room within the rigid system to begin to ask "why, " "what if" and human questions: *Why should* 357×892 *and* 892×357 *give the same answer?*

Who chose the number "360" for the number of degrees in a circle and why that number?

Really, why is negative times negative positive?

Is it better to use my 30% discount first and then apply sales tax, or compute sales tax first and then apply the 30% discount?

Why are the expressions ax^2+bx+c *called "quadratic"? I don't see the number four.*

Why do logarithms convert multiplications into additions and why did Napier want this back in the year 1600?

There is no doubt that one needs to develop facility and ease with the basic skills of mathematics so that one is not always stumbling over small matters. But the richness and true utility of the subject comes from its conceptual structure – the problem solving and analytic tools it develops and the power of the intellectual penetration it promotes. (And as a mathematician, I would add the shocking beauty and poetry of mathematics offers its own value and inspiring reward!)

The Common Core State Standards in Mathematics (have a look at illustrativemathematics.org for an approachable presentation of them) are trying to bring mathematical thinking, over rote doing, into the classroom. The eight Practice Standards, at the very least, address this. Matters are directly shifting and there is encouragement to bring mindful reflection into the student mathematics experience.

But, in practice, the typical classroom is severely pressed by the weight of mandated tests and exams. ("There simply isn't time to play with ideas.") Teachers practice their art as best they can within the parameters given, and students, by and large, perform the exactly appropriate skills needed to survive and be rewarded by the system: don't question, memorise and just do.
So … *Where is that wiggle room?*

In this essay I offer some personal thoughts aimed right at the toughest, most rigid, nugget within the system:

Where is the wiggle room in the pressured experiences of classroom tests and exams?

[For my thoughts on opening up the joyful humanness of mathematics with the curriculum content itself, wander through my site www.jamestanton.com.]

SOME STRUCTURAL THOUGHTS:

For starters, we can question what a test or exam should even look like.

Do tests need to be timed? To what extent does it matter how long a student takes to complete a set of problems?

Can some quizzes be done in pairs?

Can some quizzes or exams be open-note exams?

How about oral tests?

Can students grade each others' work?

Can students be asked to submit possible questions for an upcoming exam? [Teaching the art of writing good questions is worthwhile pursuit!]

Why give numerical scores in math? Why not give letter grades based on thinking, clarity of expression, and so on?

I routinely gave two types of quizzes that shocked a number of mathematics educators when they learned I did this:

A 100% Packet is a large, involved problem set students complete over a period of weeks. Students must earn a perfect score of 100%! They may use their notes, they may consult with me for guidance, and they are expected to hand in the packet multiple times for partial grading and feedback. They must keep working at all the problems until they have a perfect score - otherwise it is a score of 0%. (I never ended up giving out a zero score by the way!)

A Quiz with all the Answers Supplied has questions with, well, their answers supplied! *A chord in a circle of radius 10 subtends an arc of 42 degrees. What is the length of the chord? [The answer is 24.3.]* Doing this reinforces the point that I am not so much interested in the numerical answer but the method towards obtaining it. (There is also the psychological comfort of knowing whether or not your answer is correct.)

CHANGING THE QUESTIONS WE ASK:

To promote conceptual understanding we can ask *meta-questions* – questions about questions – and questions that force students to step outside of the question. Here are eight categories of these things. (I am sure you can come up with more!) I supply concrete examples to illustrate what I mean by each of them. Be gentle if you start giving questions like these to your students: they can be a bit of a shock. They are outside of the familiar rote-skills approach to learning and doing.

1: SPOT THE ERROR

Question: In answering the question: XXXXXXX Lizzy wrote: YYYYYYYYY.

Why did the teacher give her a score of only 3/5 for her response?

Question: A student writes in his homework: $6x = 18 = x = 3$. What does this actually say? What do you think the student was trying to say?

Another student writes in her homework:
$$\frac{3(x+2)}{5} = \frac{3x+6}{5} = \frac{3}{5}x + \frac{6}{5}.$$
Is this saying something reasonable?

Question : A student was asked to expand $x(x-2)$ and wrote $x = 0,\ 2$. What do you think the student was doing? Any advice?

2: CLASSIC ERRORS HEAD-ON

Battling with how hard it is to force something that is generally false to work cements the idea that it is generally false!

Question: Many students write:

$$(a+b)^2 = a^2 + b^2 .$$

a) Choose some specific values for a and b to show that this is not true in general.

b) Find a value for a and a value for b for which, by luck, $(a+b)^2 = a^2 + b^2$ happens to hold.

c) Use algebra to find <u>all</u> values for a and b for which $(a+b)^2 = a^2 + b^2$ happens to hold.

One can apply this type of question to many of the classic algebra errors.

$\sqrt{a^2+b^2} = a+b$ $\qquad$ $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$

$3 \cdot \dfrac{a}{b} = \dfrac{3a}{3b}$ $\qquad$ $\dfrac{2x+y}{2z} = \dfrac{x+y}{z}$

Question: The height of a star at an angle of elevation 30° is the same as the height of a start at 150°. Thus the following is true:

$$\sin\left(30^\circ\right) = \sin\left(150^\circ\right)$$

Lulu says: *Divide both sides by sin and get* $30 = 150$.

$$\cancel{\sin}\left(30^\circ\right) = \cancel{\sin}\left(150^\circ\right)$$
$$30 = 150$$

But clearly 30 does not equal 150!

How would you explain to Lulu that what she is doing is wrong?

Question: Come up with a scenario for which "$\frac{1}{2}$ plus $\frac{1}{3}$" could legitimately have the answer $\frac{2}{5}$.

3: EXPLAIN

Question: Joyce says that the value of $5^{\log_5 37}$ is obvious if you think about it. And she is right!

What is the value of this quantity and how would you explain to Quentin, who doesn't "get it," why the answer is what it is?

Question: Nervous Nelly, who prefers to memorise "rules" for mathematics, was once told that multiplying an inequality by a negative number "flips" the inequality.

For example, if $C < D$ then $-C > -D$. And if $x < -3$, then $-2x > 6$.

a) Has she memorized a correct rule?
b) Nelly admits she does not understand why the rule she memorized is true. How would you explain its validity to her?

Question: a) Work out $\frac{12}{15} \div \frac{3}{5}$ and show that it equals $\frac{4}{3}$.

b) Now notice that

$$12 \div 3 = 4$$
$$15 \div 5 = 3$$

and

$$\frac{12}{15} \div \frac{3}{5} = \frac{4}{3}$$

Is this a coincidence?

Or does $\frac{a}{b} \div \frac{c}{d}$ always equal $\frac{a \div c}{b \div d}$?

4: THINK BEFORE YOU LEAP!

Question: For each of the following describe an easy way to compute the answer without a calculator. Either describe your method in words or write a line of arithmetic that illustrates your way of proceeding.

a) 82×5 b) $35\times 35\times 40$

c) 7×16 d) 198×32

e) $87\cdot 903+13\cdot 903+17$

f) $196-37$ g) $817-69$

h) 621 divided by 5

i) 15% of 62 j) $\frac{13}{66}\cdot\frac{33}{28}\cdot\frac{7}{13}$

k) $603\div 97$ l) $813\div 198$

Question:
Here are four quadratic equations:

(A) $y=4(x-3)(x-7)$

(B) $y=3(x-2)^2+6$

(C) $y=2x^2-4x+8$

(D) $y=x^2+x(x-3)$

i) For which equation would it be easiest to answer the question: *What is the vertex of the quadratic?*

ii) For which equation would it be easiest to answer the question: *Where does the quadratic cross the x-axis?*

iii) For which equation would it be easiest to answer the question: *What is the smallest value the quadratic adopts?*

iv) For which equation would it be easiest to answer the question: *What is the line of symmetry of the quadratic?*

v) For which equation would it be easiest to answer the question: *What is the y-intercept of the quadratic?*

Question: Which of the following problems is <u>not</u> easy to work out in your head?

$23\times 37-13\times 37$

$27\cdot 153+73\cdot 153$

$3(7)+87(7)$

$105(105)-95(105)$

$17\times 13+13\times 3$

$34\times 7+34\times 6$

Question: Compute
$(7-18)(8-18)(9-18)(10-18)\cdots\cdots(25-18)(26-18)$

Question: Which of the following statements seem they could be true? Which are definitely wrong? (Answer this question without actually computing the products. Not one of them is actually correct! This is an exercise in estimation only.)

$999\times 31=30999$

$12\times 198=1996$

$106\times 213=206,816$

$9458\times 9786=192837261748$

$19990\times 4=76987$

Question: How big an answer do you expect computing 19×998?

Question: Quickly … solve:

a) $(x-2)(x-14)(x-22)=0$

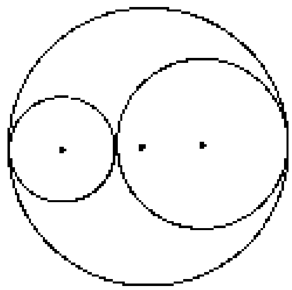
b) $(x+1)^2=25$

Question: A parabola passes through the points $(2,5)$, $(3,-6)$ and $(10,5)$. What is the x-coordinate of its vertex?

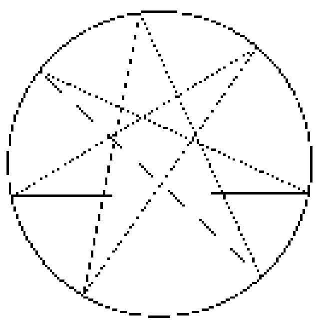
Question: Find the area of a triangle with side lengths 9 inches, 8 inches and 19 inches.

5: DISCOVER AND EXPLORE

Question: The centers of all the circles in this picture are collinear. Discover and explain something interesting about the circumferences of these circles



Question: Discover and explain something interesting about the angles of a pointed star drawn inside a circle as shown.



Question: Playing on a calculator Pandi noticed that 2^{46}, 2^{56} and 2^{76} each begin with a seven.

a) Find the next few powers of two that begin with a seven.
b) Explain why the pattern you are seeing stops.

Question: Sketch $y = x^{\frac{1}{\ln x}}$ on a calculator. Explain!

6: JOLT!

Bring in unexpected connections to provoke clarity!

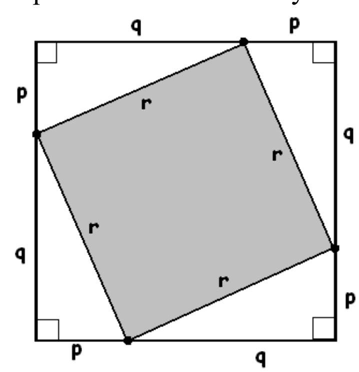
Question: a) Compute $(x^6 + x^5 + 5x^4 + 5x^3 + 9x^2 + 5x + 2) \div (x^2 + x + 2)$

b) Put $x = 10$ into the problem of part a). What grade-five division problem have you just solved? Does your answer seem to be correct?

People forget in an algebra course that x can actually be a number!

Question:
a) Show that $x^5 - 1$ is divisible by $x - 1$.
b) Is $2^{100} - 1$ prime?

Question: Compute the area of the central shaded square two different ways:



Question: Consider the differential equation $\frac{dy}{dx} = iy$ with $y(0) = 1$.

a) What would be the standard solution to this differential equation?
b) Show that $y = \cos x + i \sin x$ is also a solution.

What might you be tempted to now say?

7: PUSH BOUNDARIES

Question:
In computing 654 + 179 Iggy writes:

$$\begin{array}{r} 654 \\ +\,179 \\ \hline \end{array}$$

7 12 13 = 833

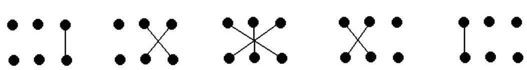
Does this represent valid mathematical thinking? Briefly explain what you guess Iggy was thinking.

Question:
a) When asked to find a fraction between $\frac{1}{12}$ and $\frac{1}{11}$, Poindexter wrote: $\frac{1}{11\frac{1}{2}}$.

Is this a mathematically valid answer?

b) Quickly write down three fractions that lie between $\frac{1}{12}$ and $\frac{1}{11}$.

Question: Vedic mathematics taught in India (and established in 1911 by Jagadguru Swami Bharati Krishna Tirthaji Maharaj) has students compute the product of two three-digit numbers as follows:



What do you think this sequence of diagrams means?

8: SOMETHING IS NOT RIGHT

These are like "spot the error" questions but require a deeper level of thought.

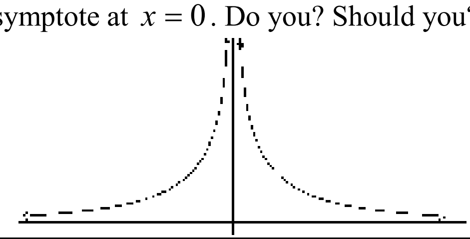
Question:
Could $56452 \times 18863 = 98611987364$ be correct?

Question: My calculator says that $\sqrt{46}$ equals 6.782329983. What can't this be correct?

Question: Could the sum of 19000 odd numbers end with a five?

Question: Explain why the formula $(a+b)^4 = a^4 + b^4 + 4ab^3 + 6ba^3 + 4a^2b^2$ cannot be right.

Question: When I plot the curve $y = \frac{200}{1+x^4}$ on my calculator I see a vertical asymptote at $x = 0$. Do you? Should you?



Question: Sketch the graphs $y = x^2$ and $y = 2^x$ simultaneously on a calculator to se they intersect twice. Is this right? Do they intersect exactly two times?

Question: Tatiana says $10! = 8542677640$. Quickly, why must she be mistaken?

TWO FINAL COMMENTS:

On Homework: *Let go of busy work for the sake of busy work.*

If your students have the appropriate self-reliance and maturity one can consider handling chapter homework problems along the following lines.

1. Answer enough questions from this list of problems to reach the point you feel you understand what is going on.

2. Of the questions you left unsolved, pick the one you feel will be the easiest to do and the one you feel will be the hardest to solve. Do them, and see if your sense of them were more or less right. Write me a sentence or two about this experience. (Be sure to let me know which two problems you worked on.)

3. Of the harder questions you did, what made the hardest ones so hard? Make up one problem that is similar to these hard questions to give to a friend, one that uses the same ideas but is actually a bit easier to solve. Share that problem with me.

A slightly longer version of this essay appears at http://www.jamestanton.com/?p=968

A Word on Words: Take note of the words we regularly use for things we make students do in the mathematics classroom:

exercise
problem
challenge
exam
worksheet

We discuss how we might go about *attacking* a problem.

And what is the standard extracurricular math activity? Joining a math team … to prepare for *competition*.

How did this draining, non-fun, combative language come to be in mathematics?

Rather than "attack a problem" can we perhaps "probe an idea"? Rather than do thirty exercises for homework (and why not add thirty push-ups to that as well?) can we engage in explorations? Can the first words that come to mind in describing "extracurricular mathematics" be joyful intellectual play?