TANTON'S TAKE ON …

THE VINCULUM

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

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What does $2+3+4$ *mean?*

This might seem like a silly question. (It means nine!) But upon a moment's reflection we might come to realize that there is a question here, not about the answer, but about the process of getting to the answer.

There are two ways to compute $2+3+4$:

Add together 2 and 3 first to get 5, and think "that answer plus 4*."*

OR

Add together 3 and 4 first, remember the answer, and think "2 *plus that answer."*

(We are keeping the numbers 2, 3 and 4 in the order they appear.)

In the 15th – and 16th- centuries, European mathematicians used a horizontal bar, called a *vinculum*, to indicate "grouping together." To demonstrate the first line of thinking they would write $\overline{2+3}+4$ giving $\overline{2+3}+4=5+4=9$. The second line of thinking would be expresses as $2+\overline{3+4}$, yielding $2+\overline{3+4}=2+7=9$.

Question: How would folk from Renaissance Europe wish you to think through each of the following?

a) $\overline{3+7}+3$ b) $\overline{3+9}-7$

c) $3+\overline{9-7}$ d) $\overline{4-1}-\overline{3-2}$

Can you make sense of these expressions with nested vinculums?

e) $\overline{2+\overline{5-1}}+3$

f) $6+\overline{18-\overline{3+\overline{\overline{7-1}+4}}}$

It is very easy to read (and in this day and age to type) expressions involving vinculums, even complicated nested vinculums. For example, can you see right away where to begin with this?

Evaluate the following for $q=5$*:*

$$\overline{\overline{7-\overline{\overline{\overline{2-\overline{3-\overline{q+2}}}+q}-3}}-\overline{q+1}}+q$$

But the vinculum is no longer used today – which I personally find sad. We prefer to use *parentheses* (also called *brackets*) to represent the grouping of symbols. For instance, the above expression is today written:

$$\left(\left(7-\left(\left(\left(2-\left(3-\left(q+2\right)\right)\right)+q\right)-3\right)\right)-\left(q+1\right)\right)+q$$

Is this easier to read? (Have we trained ourselves to think it is easy?)

Comment: When teaching students how to interpret nested parentheses, I've seen educators do the following:

First underline the innermost parentheses:

$$\left(\left(7-\left(\left(\left(2-\left(3-\left(\underline{q+2}\right)\right)\right)+q\right)-3\right)\right)-\left(q+1\right)\right)+q$$

Then the second-inner most parentheses:

$$\left(\left(7-\left(\left(\left(\left(2-\left(\underline{3-\left(\underline{q+2}\right)}\right)\right)\right)+q\right)-3\right)\right)-(q+1)\right)+q$$

And the next

$$\left(\left(7-\left(\left(\left(\left(\underline{2-\left(\underline{3-\left(\underline{q+2}\right)}\right)}\right)\right)+q\right)-3\right)\right)-(q+1)\right)+q$$

And so on

Looks like a return to the vinculum to me!

Why the change? In the 15th century the printing press was invented and for the first time books could be printed and produced in mass. To print a page of text one arranged small printing blocks in rows on a large tray, one tile for each letter, number and symbol. The blocks were then covered with ink and a sheet of paper was pressed against them.

The vinculum in mathematics texts presented a problem: it is a symbol that does not lie in the same row as the text and the numbers. One could first print the page without the vinculum, and then reset the tray of tiles for just the vinculums set in the correct half-level positions for a second round of printing of that same page. But this is mighty inconvenient! So to obviate this difficulty, mathematicians settled upon a notation for grouping symbols that remains in the same line as the text. They used parentheses.

Exercise: Rewrite each of the following using parentheses.

a) $\overline{4-1}-\overline{3-2}$

b) $\overline{2+\overline{5-1}}+3$

c) $6+\overline{18-\overline{3+\overline{\overline{7-1}+4}}}$

Here are some expressions in modern parentheses notation. How would they have been written in Renaissance Europe?

d) $\left(4+\left(4+2\right)\right)+1$

e) $8-\left(\left(4-2\right)+1\right)$

f) $\left(\left(3-\left(\left(4+1\right)-2\right)\right)+5\right)+3$

Exercise: Ethel says that there are three "rules" about parentheses:

RULE ONE*:* *If a single set of parentheses appear in an expression, compute the value of the quantity inside the parentheses first and then move on from there.*

e.g. $2+\left(1+2\right)=2+3=5$

e.g. $\left(4+3\right)-8=7-8=-1$

RULE TWO: *If there are parentheses within parentheses, compute the innermost parentheses first.*

e.g. $4+\left(\left(6+3\right)+1\right)=4+\left(9+1\right)=4+10=14$

RULE THREE: *If there are "equally inner" parentheses, compute them simultaneously.*

e.g. $2+\left(\left(4-5\right)+\left(2+3\right)\right)=2+\left(-1+5\right)=2+4=6$

a) Rewrite each of the five examples given in terms of the vinculum.

b) From your understanding of the vinculum, do these rules appear to be correct?

Here is a great activity.

COUNTING VINCULUMS

Consider the sum $1+2+3$.

There are **two** ways to place a vinculum in this sum (keeping the order of the terms the same) so that one is summing only two numbers at a time: $\overline{1+2}+3=3+3=6$ and $1+\overline{2+3}=1+5=6$.

Consider the sum $1+2+3+4$.

There are **five** ways to place vinculums so that only two numbers are ever being added at a time:

$$\overline{\overline{1+2}+3}+4=\overline{3+3}+4=6+4=10$$

$$\overline{1+\overline{2+3}}+4=\overline{1+5}+4=6+4=10$$

$$1+\overline{\overline{2+3}+4}=1+\overline{5+4}=1+9=10$$

$$1+\overline{2+\overline{3+4}}=1+\overline{2+7}=1+9=10$$

$$\overline{1+2}+\overline{3+4}=3+7=10$$

a) There are **fourteen** ways to place vinculums about the sum $1+2+3+4+5$. List all 14 ways!

There is really only **one** way to place a vinculum over the sum $1+2$.

So far we have the "vinculum numbers":

1 2 5 14

b) Begin listing all the ways to place vinculums over a sum of six numbers. Do this in a systematic manner and see if you can explain why the count of possibilities is $1\times 14+1\times 5+2\times 2+5\times 1+14\times 1=42$.

c) What is the next vinculum number after this? And the next?

d) Is there an explicit formula for the n th vinculum number?

THE VINCULUM HAS NOT GONE AWAY

The vinculum makes its appearance still today in mathematics. You have no doubt used it without realizing it:

1. ROOTS. The symbol √, called a *radix*, means "square root" and when accompanied with a vinculum it indicates the quantity whose square root is to be taken. For example, √9 + 16 equals 19 whereas $\sqrt{9+16}$ is 5. Here the vinculum dictates that 9 and 16 are to be treated as one group.

2. LINE SEGMENTS. In geometry if two points A and B are to be "grouped" as one entity as a line segment, we denote that segment $\overline{AB}$. (No doubt that seventeenth-century Italian mathematician Bonaventure Cavalieri who first used this notation thought it convenient that the vinculum itself looks like a line segment.)

3. REPEATED DECIMALS. An infinitely long repeating decimal is often described with the aid of a vinculum. For example, $0.\overline{824}$ indicates that "824" is to be regarded as one group that is repeated infinitely often: $0.\overline{824}=0.824824824\cdots$.

4. DIVISON. The division symbol $\div$, itself called an *obelus*, is composed of a vinculum and two dots to represent a fraction. Even for fractions themselves, the vinculum we use in their notation represents grouping. For example, in the expression:

$$\frac{4+6}{2}$$

the vinculum indicates that $4+6$ is the entity to be computed as its own group.

$$\frac{4+6}{2}=\frac{10}{2}=5$$

As another example, in:

$$\frac{7+5}{3+1}$$

the vinculum indicates that $7+5$ and $3+1$ are each their own entities to be computed first, thus:

$$\frac{7+5}{3+1}=\frac{12}{4}=3.$$

And the nested vinculums tell us how to untangle:

$$\frac{16+\frac{10}{4+1}}{3+7}$$

We have

$$\frac{16+\frac{10}{4+1}}{3+7}=\frac{16+\frac{10}{5}}{3+7}=\frac{16+2}{3+7}=\frac{18}{10}=\frac{9}{5}.$$

Question: Untangle each of the following:

a) $$\frac{\frac{3+4}{7}+\frac{10}{1+1}}{2+\frac{10+22}{10-2}}$$

b) $$3+\cfrac{\cfrac{4+1}{5+\cfrac{6}{2+1}}-\cfrac{12-7}{3+\cfrac{16}{8-4}}}{2+\cfrac{23-6}{23-\cfrac{1}{1-\cfrac{787}{987+765}}}}$$

(This one is not as hard as it looks!)

PEDAGOGY:
From Arithmetic to Algebra

Many students find the transition from arithmetic (working with actual numbers) to algebra (working with unknown numbers represented by symbols) quite difficult. For example, computing:

$$\frac{4+6}{2}$$

as $\frac{10}{2}=5$ (keeping the vinculum in mind) usually offers no conceptual difficulty, but manipulating:

$$\frac{x+y}{2}$$

often does. It is tempting to say that $\frac{x+y}{2}$ equals $\frac{x}{2}+y$. (It never seems tempting to say, however, that $\frac{x+y}{2}$ equals $x+\frac{y}{2}$!)

The reason for this conceptual difficulty is that, *in algebra, we can't actually perform the operation the vinculum is asking us to first do!* Yet we are being asked to do something with the expression nonetheless.

There are three approaches that might help.

1. **Try some actual numbers.**
"I am tempted to write down $\frac{x+y}{2}=\frac{x}{2}+y$. Can this be right?"

Let's test this with numbers. Is $\frac{4+6}{2}$ equal to $\frac{4}{2}+6$? No! The first is $\frac{10}{2}=5$ and the second is $2+6=8$. This guess is wrong and so we need to think through to something else.

WARNING: The "try some numbers" technique is a wee bit dangerous. If the computations turn out not to match, then you know, for sure, the identity you are hoping to be true is for sure wrong. But calculations that do match should leave you wondering: *Did my answers match because the equation I have is always true, or because I happened to choose numbers for which it worked only by coincidence?* For instance, to test if squaring a number and doubling a number are algebraically the same (is x^2 algebraically equivalent to $2x$?) I might be fooled by testing $x=2$ and $x=0$.

Exercise: Find values of a and b for which each of these common student mistakes happen to be true:

$$(a+b)^2=a^2+b^2$$
$$\sqrt{a+b}=\sqrt{a}+\sqrt{b}$$
$$\sin(a+b)=\sin(a)+\sin(b)$$
$$2^{a+b}=2^a+2^b$$
$$\frac{1}{a+b}=\frac{1}{a}+\frac{1}{b}$$

2. **Think in terms of units**
In $\frac{4+6}{2}$ the vinculum is telling us to think of "$4+6$" as one group that is to be divided by two. For example, we might have 4 apples and 6 bananas, but as a single group this is 10 pieces of fruit. Thus $\frac{4+6}{2}$ represents an entire group that is to be divided by two. It thus yields 5 pieces of fruit. (In fact, 2 apples and 3 bananas.)

In $\frac{x+y}{2}$ the vinculum is telling us to think of "$x+y$" as one group to be divided by two. So instead of thinking of x apples and y bananas, think $x+y$ pieces of fruit. Dividing by two yields half the fruit – in fact $\frac{x}{2}$ apples and $\frac{y}{2}$ bananas. (And to be clear, the option $\frac{x+y}{2}=\frac{x}{2}+y$ is wrong because it is not yielding half the fruit – this is half the apples but all of the bananas.)

WARNING: This approach is dangerous if students take the notion of units too literally! For example, with 4 apples and 6 bananas, we can take half the fruit by selecting all the apples and just one banana, suggesting that $\frac{x+y}{2}=x+1$. (And this equation happens to be true for $x=4$ and $y=6$.) Who is dictating how the fruit is divided? What distribution scheme should we assume? In more complicated situations, translating to a literal model can be more confusing than helpful.

3. **All division is multiplication!**
Diving by two is equivalent to multiplying by one half: $\frac{A}{2}\equiv\frac{1}{2}\cdot A$. (Is this obvious?)
So let's obviate all our worries with division problems by rewriting them as multiplication problems!

Now $\frac{x+y}{2}$, noting the vinculum, represents the group $x+y$ to be divided by two. This is equivalent to multiplying the group by one-half instead.

$$\frac{x+y}{2}=\frac{1}{2}\cdot(x+y)$$

Distributing gives:

$$\frac{1}{2}(x+y)=\frac{1}{2}\cdot x+\frac{1}{2}\cdot y$$

And putting it all together yields:

$$\frac{x+y}{2}=\frac{x}{2}+\frac{y}{2}.$$

This approach helps with the classic temptation to interpret $\frac{2a+b}{2}$ as $a+b$. ("The twos cancel!")

$$\begin{aligned}\frac{2a+b}{2}&=\frac{1}{2}(2a+b)\\&=\frac{2a}{2}+\frac{b}{2}\\&=a+\frac{b}{2}\end{aligned}$$

It also helps for situations with a complicated denominator.

Example: Rewrite $\frac{6a+x+2y}{x+2}$.

Answer: The vinculum tells us to think of two sets of groups. We have the group of objects "$6a+x+2y$" to be divided by the group "$x+2$." Dividing by $x+2$ is the same as multiplying by the fraction $\frac{1}{x+2}$, so our expression can be written instead as:

$$\frac{1}{x+2}(6a+x+2y).$$

Distributing, we have:

$$\frac{1}{x+2}\cdot 6a+\frac{1}{x+2}\cdot x+\frac{1}{x+2}\cdot 2y$$

Taking it slowly, making sure we are clear on our fraction multiplication, we can rewrite this as:

$$\frac{1}{x+2}\cdot\frac{6a}{1}+\frac{1}{x+2}\cdot\frac{x}{1}+\frac{1}{x+2}\cdot\frac{2y}{1}$$

Now we see:

$$\frac{6a}{x+2}+\frac{x}{x+2}+\frac{2y}{x+2}.$$

Comment: I think the original expression looked simpler and friendlier!

Explaining the Numerator/Denominator Imbalance:

"Distributing over a numerator" in fractional expressions is a valid algebraic operation. For example,

$$\frac{x+a}{x+b}=\frac{x}{x+b}+\frac{a}{x+b}$$

is correct. But "distributing over a denominator" is not. Trying some numbers shows that the following is not true in general.

$$\frac{x+a}{x+b}=\frac{x+a}{x}+\frac{x+a}{b}$$

This imbalance, of sorts, is confusing to many students. Returning to a numerical example, with units, can help sort out the confusion.

Consider, for instance, the fraction $\frac{4+6}{2+3}$.

According to the vinculum, this represents the value $4+6=10$ to be divided by the value $2+3=5$, and so the fraction is just the number $\frac{10}{5}=2$.

But fractions can be interpreted as answers to sharing problems (that is, division problems). If we think in terms of units, $\frac{4+6}{2+3}$ perhaps represents 4 apples and 6 bananas (grouped together as 10 pieces of fruit via the vinculum) to be shared among 2 boys and 3 girls (grouped together as 5 children). The answer is two pieces of fruit per child.

Consider the expression given by "distributing over the numerator."

$$\frac{4}{2+3}+\frac{6}{2+3}$$

Is this sharing the fruit correctly? YES! It shares the four apples among the five children and adds to that the result of sharing the six bananas among the five children. The ten pieces of fruit are indeed being distributed equally among all five.

Consider the expression given by "distributing over the denominator:"

$$\frac{4+6}{2}+\frac{4+6}{3}$$

This is asking us to distribute ten pieces of fruit among the two boys and distribute ten pieces fruit among the three girls. That is too much fruit being shared out in disproportionate ways! It is clearly NOT equivalent to $\frac{10}{5}$.

This example illustrates that in sharing goods equally among people, it is valid to separate the goods into sections and share those sections equally among all the folk one section at a time. It is not valid to separate the people into groups, and then share ALL the goods to each of those groups over and over again.

But of course… The true reason for this perceived imbalance is the mathematics. The distributive rule holds only in one setting and not the other:

$\frac{1}{x+b}(x+a)=\frac{1}{x+b}\cdot x+\frac{1}{x+b}\cdot a$ is valid.

$\frac{1}{x+b}(x+a)=\frac{1}{x}\cdot(x+a)+\frac{1}{b}\cdot(x+a)$ is not.

In my algebra classes we spend a lot of time rewriting *all* division problems as multiplication problems (not being afraid of fractions) just to be clear on what the mathematics insists we do.

Follow Ups: This essay is based on the work that appears in *THINKING MATHEMATICS! Vol 1, Chapter 1* and *MATHEMATICAL THINKING, Chapter 13* (available at www.lulu.com).

I have a video "**Bring back the Vinculum**" at www.jamestanton.com/?p=1258 and the vinculum exploration described here is solved at www.jamstanton.com/?p=1079.

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