TANTON'S TAKE ON …

TROUBLESOME ZERO

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

DECEMBER 2012

In response to the gasoline crisis after superstorm Sandy Mayor Bloomberg announced:

Those with license plates ending in an even number, or the number zero, will be able to buy gas or diesel only on even-numbered days, such as Saturday,Nov. 10.

Math bloggers and members of the math twitterverse were aghast that Bloomberg felt the need to set zero as an exceptional case.

This is not a first example of zero-confusion. During the smog crisis of 1977 Parisian authorities enforced the rule that on odd numbered days only cars with license plates ending with an odd number could be on the roads, cars ending with an even number on even numbered days. It is said that owners of cars with plates ending with 00 or 000 flaunted public ignorance by driving each and every day nonetheless. Parisian officers were unclear on the status of zero, is it even or odd ?

Aside Question: Are these rules fair in the long run? How many odd-numbered days are there in a year? How many even-numbered days?

"Elementary" questions in mathematics are often superbly difficult. As one grows in sophistication as a mathematical thinker, one's mind naturally starts coming back to basic ideas, questioning them, realizing that that there is actually something to question. (If only our curriculum weren't so focused on trudging forward, forward, forward, more, more, more, do, do, do.)

"Zero" is a truly thorny concept. Mayor Bloomberg was right to be explicit about the role of zero in any set of instructions. The public's general confusion over "zero" says a great deal about the number.

Qestion: Beginning mathematics starts with the counting numbers: $1, 2, 3, 4, \ldots$. These are the numbers that, well, count things: one apple, five gnus, 18 misuses of the semi-colon, for example.

Should zero be on this list? Is zero a counting number? If I say "There are zero giraffes in this room" am I counting zero giraffes or just observing a lack of giraffes? What do you think?

Here are seven "elementary" questions about zero. What are your personal thoughts on each? How would you respond to a student who asks one of these questions? (Start by lavishing much praise on to the lass or lad for being a good deep thinker!)

Mull on these issues for yourself before reading on. Do you have personally satisfying answers to each? ("It just is" isn't very satisfying!) Really mull. Savor the mulling!

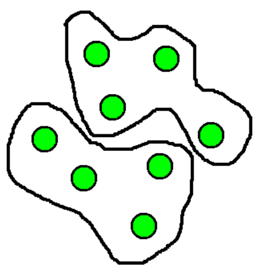
Thorny Question 1.
Okay. So … Is zero even or odd? Both or neither?

Thorny Question 2.
Why can't one divide by zero? For example, what is wrong with $\frac{5}{0}$? Is $\frac{0}{0}$ okay?

Thorny Question 3.
Is -0 the same as 0?
Thorny Question 4.
Does 0 have a square root?
Thorny Question 5.
Why is $0!$ equal to one?
Thorny Question 6.
Why is 2^0 equal to one?
Thorny Question 7.
What is the value of 0^0?

SOME THOUGHTS…

Thorny Thoughts 1. In the world of counting numbers we say that a number is *even* if we can split a group of that many objects into two piles of equal size (with no splitting of objects themselves!). Otherwise we call the number *odd*. For example, this picture of eight dots splitting into two equal piles shows that 8 is even.



Trying this with a picture of nine dots will yield one dot left over. So 9 is odd.

Question: In the number system of fractions does 9 pass the test of being even?

Suppose we consider zero as part of the system of counting numbers. Can a picture of zero dots be split into two equal piles with no dots left over? Yes! Split it into two empty piles! In this sense, zero is even.

Question: Does this pictorial mode of thinking make the following "rules" of even/odd arithmetic clear?

$$even + even = even$$
$$even + odd = odd$$
$$odd + even = odd$$
$$odd + odd = even$$

A more mathematical approach: For those who like formal mathematical jargon: A counting number n is said to be *even* if there is another <u>counting</u> number a such that $n = 2a$. A counting number n is said to be *odd* is there is another <u>counting</u> number b so that $n = 2b+1$. (This is a formalization of the picture idea.)

Does zero follow the definition of being even? Yes. $0 = 2a$ for the choice $a = 0$! (Oooh! This has assumed we have agreed to consider zero a counting number!)

Question: Suppose we have the audacity to extend the definition of evenness and oddness to integers, $-3,-2,-1,0,1,2,3,\ldots$. In this setting, is -13 even or odd? (Can you find an integer b so that $-13 = 2b+1$?)

Question: Sammy yells "Twenty-four pages have been ripped out of my algebra book." Ayelet replies: "I bet the sum of all the missing page numbers is even." She is right! How did Ayelet know?

Thorny Thoughts 2. One can check the answer to a division problem by looking at its reverse: multiplication. For example, $\frac{20}{5} = 4$ is right because indeed 4×5 is 20. One sees that $\frac{7}{3} = 2$ is wrong because it does not pass the multiplication check: $2 \times 3 \neq 7$.

Consider $\frac{5}{0}$. If I say that the answer is three, $\frac{5}{0} = 3$, I would be proved wrong by noting that $3 \times 0 \neq 5$. If, instead, I say the answer is $-17\frac{3}{4}$, I would be proved wrong again: $-17\frac{3}{4} \times 0 \neq 5$. In fact there is no number x for which $\frac{5}{0} = x$ passes the multiplication check: $x \times 0$ is always zero, never 5, and so there is no number that can be the answer to $\frac{5}{0}$. In the same way, there can be no answer to $7/0$ or $62/0$ or $\pi/0$, and so on.

On the other hand, dividing zero by zero is fine! It equals thirty-six: $\frac{0}{0}=36$. This is right because 36×0 does equal 0. Do you believe me?

The problem with $0/0$ is that rather than having no possible answer in this arithmetic it has too many answers: $\frac{0}{0}=\sqrt{5}$ passes the check (because $\sqrt{5}\times 0=0$ is true) and $\frac{0}{0}=8$ passes the check (because $8\times 0=0$), and so on.

Division by zero is not well defined in this context and so is deemed best to avoid. (Upper school calculus, on the other hand, is all about living dangerously and heading to the limit of dividing by zero. How fun!)

Question: In our discussions to questions 1 and 2 we assumed the "property of zero," that $a\times 0=0$ for all numbers a. Do you believe this rule? Does it feel right? On the level of very basic counting at least does it feel proper to say, for instance, that 5×0, "five groups of nothing," should be nothing?

Thorny Thoughts 3. We usually think of a negative quantity as that which cancels a positive quantity. That is, we define "$-a$" to be a number, which when added to a, gives zero: $(-a)+a=0$.

For example -3 cancels three: $-3+3=0$, and -1.87 cancels 1.87.

Aside: So this means $-(-5)$ is a number, which, when added to -5, gives zero. But 5 fits that bill! So $-(-5)=5$.

Question: What's $-(-(-5))$?

What then is -0? It is a number which, when added to 0, gives zero. Well 0 itself works: $0+0=0$. So $-0=0$

Comment: Many grade-school curricula give models for negative numbers in which "$-$" is interpreted as "opposite." -5 is the opposite of five dollars credit and so represents five dollars of debt. Or -3 is the opposite three piles and so is three holes. In these models -0 is the opposite of no credit or debt (which still no credit or debt) or the opposite of no piles or holes (the opposite of a flat sand bed is still another flat sand bed), and so on. These models suggest too that $-0=0$.

Question: Our definition of $-a$ said "a number" which when added to a gives zero. Could there be more than one number that works? Could this definition fail to be well defined?

AN EIGHTH THORNY QUESTION:
Is zero positive or negative? Explain.

Thorny Thoughts 4. The name "square root" is very geometric. The root feature of a square (its base feature, its primary feature) is its side-length, and the square root of a positive number is the side-length of a physical square whose area is that number.

For example, the side-length of a square of area 16 is 4, $\sqrt{16}=4$, and the side-length of a square of area 2 is some number close to 1.414, $\sqrt{2}\approx 1.414$.

This is the true, ancient Greek definition of square root. It is a geometric notion and only applies to actual geometric objects. (And so the Greeks would never write $\sqrt{9}=\pm 3$, for instance. Do you see why? Mathematicians will not write this either, honoring the geometric meaning of the symbol "$\sqrt{\ }$.")

So what is $\sqrt{0}$? It is the side-length of a square of no area. Not much of a square! So is this a meaningful notion? Maybe not.

But perhaps we can argue that a square of no area has no side-length too? In which case we could say $\sqrt{0}=0$. From a geometric viewpoint, asking for the square root of zero might be do-able after all but it is asking for a bit of a philosophical stretch.

From an arithmetical point of view, however, taking square root of zero might be considered woe free. Here, by $\sqrt{a}$ we simply mean a number such that $\sqrt{a}\times\sqrt{a}$ equals a. Since 0×0 equals 0, we have that 0 itself fits the bill of being $\sqrt{0}$.

Most folk choose to accept this arithmetic approach and do say that $\sqrt{0}=0$.

Question: What would the Greek scholars of ancient times say about $\sqrt{-4}$? What might someone who studies the arithmetic of the real numbers say about it?

Question: Here are two questions:

i) Find $\sqrt{9}$.

ii) Solve $x^2=9$.

One of them has one answer. The other has two answers. Do you understand why?

Question: Is $0\times 0=0$ true? What are zero groups of nothing? If I have "no nothing" it must be because I have something????

Thorny Thoughts 5: For a counting number N, the factorial $N!$ is defined to be the product of all the numbers from 1 up to N. For example, $5!=1\times 2\times 3\times 4\times 5=120$. $1!=1$ just squeaks fitting into this definition, but $0!$, the "product of all the numbers from one up to zero" does not!

Factorials arise repeatedly in the study of counting arrangements and selections of objects. For example, not worrying about order, there are ${}_N C_k=\dfrac{N!}{k!(N-k)!}$ ways to select k people for a committee from a pool of N folk, unless $k=0$, in which case there is one way to select no people (just sit back and do nothing) or $k=N$, in which case there is one way to select everyone. (Both $k=0$ and $k=N$ produce the term $0!$ in the formula.)

It is very annoying to have to point out the extreme cases every single time when discussing a counting formula. But this annoyance is easily swept away if we all agree, as a general convenience, to set $0!$ to equal one. And mathematicians have all agreed to do this.

Question: There is no means to select 11 objects from 10. What does the ${}_N C_k$ formula have to say about $(-1)!$?

Comment: I have a considerable amount to say about the topic of permutation and combinations, factorials, Pascal's triangle and more. See:

www.jamestanton.com/?p=659 .

I am on a personal mission to eradicate the terms "permutations" and "combinations" from school curricula, setting students and teachers free from the brain-ache of "Does order matter?"

Thorny Thoughts 6: If a is a positive counting number we define 2^a to mean the product of a twos. For example, 2^3 is the product of three 2 s, $2^3=2\times 2\times 2=8$, and $2^{10}=2\times2\times2\times2\times2\times2\times2\times2\times2\times2=1024$

Interpreting 2^1 might be fine in this context ($2^1=2$, but is this really the product of one two?) but certainly 2^0 has no meaning here (the product of no twos is … ?)

Some people like patterns and say that from:

$$2^4=16$$
$$2^3=8$$
$$2^2=4$$
$$2^1=2$$
$$2^0=?$$

it makes sense to set $2^1=2$ and $2^0=1$ (and $2^{-1}=0.5$ and $2^{-2}=0.25$, and so on.)

Question: According to the pattern, what then is $2^{0.5}$? Hmm.

Mathematicians develop a study of new number system by asking: *How would we like the arithmetic of this system to behave?* For example, it is clear that $2^3 \times 2^5 = 2^8$ (the product of three 2s multiplied by the product of five 2s gives a product of eight 2s), and in general

$$2^a \cdot 2^b = 2^{a+b}$$

is clearly valid whenever a and b are counting numbers.

Here's the question: Does this "rule" feel so valid and right that we would like it true for all types of numbers a and b?

If we play the game that this rule is to hold no matter what, there will be consequences. Writing $2^0 \cdot 2^3 = 2^3$, for example, which reads as $2^0 \times 8 = 8$, shows us we have no choice but to set $2^0 = 1$ in this game.

Question: Use the same rule to explain we are forced to deem $2^1 = 2$. Also use the rule to explain why $2^{-1} = 1/2$ and $2^{\frac{1}{2}} = \sqrt{2}$.

Comment: No one is saying that you must agree to $2^a \cdot 2^b = 2^{a+b}$ holding for non-counting numbers! Maybe you can devise a new type of arithmetic with its own, and different, logical consequences. Maybe in your arithmetical system 2^0 is not one! (But be aware, the mathematics studied in the K-12 curriculum plays the game of assuming this rule to always hold.)

Thorny Thoughts 7: Just like 2^0 , each of 0.1^0 and 0.01^0 and 0.001^0 and so on, is one. The sequence $0.1^0, 0.01^0, 0.001^0, \ldots$ is always 1 and is clearly approaching 0^0 . So $0^0 = 1$.

On the other hand, $0^{0.1}$ is the tenth-root of zero, which is zero. And $0^{0.01}$ is the hundredth root of zero, which is zero. And $0^{0.001}$ it the thousandth root of zero, which is zero. And so on. The sequence $0^{0.1}$, $0^{0.01}$, $0^{0.001}$, … is always zero, and is clearly approaching 0^0. Thus $0^0 = 0$.

Question: On a calculator work out $0.1^{0.1}$ and $0.01^{0.01}$ and $0.001^{0.001}$, and so on. What does this suggest about the value of 0^0?

Actually both answers $0^0 = 1$ and $0^0 = 0$ are wrong. The true answer is: $0^0 = 36$. To see why, go back to the basic rule of exponents we choose to believe:

$$0^a 0^b = 0^{a+b}.$$

We have then, for example, $0^0 \times 0^3 = 0^3$. Now $0^3 = 0 \times 0 \times 0 = 0$, $0^0 = 36$ and $0^0 \times 0^3 = 36 \times 0 = 0 = 0^3$ all works!

Question: On a calculator sketch $y = x^x$ for $x > 0$. (Can you put negative inputs into the formula x^x? Negative fractions?) What does your graphing calculator want 0^0 to be?

Question: Actually, why not just type 0^0 into a calculator? What answer does it give?

So, what do you think is the answer to the question "What is 0^0?"

PEDAGOGICAL COMMENT:
You may have noticed that this is not a one page essay!

If math were dictatorial, then the answer section to the thorny questions would be very short, something along the lines …

ANSWERS:
1. It is even.
2. Division by zero is not permitted.
3. -0 is the same as 0.
4. $\sqrt{0} = 0$
5. It just is.
6. It just is.
7. It is undefined.

MEMORISE THESE. Quiz tomorrow.

Mathematics is not at all dictatorial – but honest. A result or study in this subject begins by examining a system and getting a good feel for it - like young students getting a feel for arithmetic through the counting numbers, or getting a feel for geometry by playing with shapes and drawing. One identifies key features that seem “right” and “natural” to the system. For example, in arithmetic, we come to feel that $a \times b = b \times a$ always holds for counting numbers (turn a rectangle of dots ninety degrees) or in geometry that angles in a triangle seem to add to half a turn.

Once mathematicians have identified some key features of a study (I hesitate to call them “rules”) they then like to play the following game:

If we choose to believe these features as always true - no matter what!- what then follows as logical consequences?

Mathematicians are painfully honest and are fully aware they are playing an “if” game and that no one is saying one need believe the rules in the first place! In fact, mathematicians also delight in finding alternative systems of arithmetic and geometry in which different rules hold. (And curiously, these alternative theories always seem to find real-world applications too. For example, in the arithmetic needed describe the mathematics of fundamental particles, quantum mechanics, the rule $ab = ba$ need not hold. In the geometry on the surface of the Earth, angles in triangles do not add to half a turn.)

My responses to the seven thorny questions are not short and definitive, but they are real. I have just as many questions in my answers as I do answers to the questions. This is true mathematics: organic, and messy, and deep, and confusing. And all that is okay. Short definite answers come only after the context of the study has been spelled out. “If I follow these rules of arithmetic, must negative times negative be positive?” The answer is then either yes or no. Somehow K-12 educators and their students are meant to a priori know what rules are being played despite the in fact they are usually never brought up for conversation and debate.

I hold all the discussions I have presented here with my students and I do not shy away from the messiness of the confusion. But at the student-level it is important to give something concrete and clear to hold on to in the end. I offer it … in a slightly obtuse way:

> “Well, most of the world wants us to accept the rule $2^a 2^b = 2^{a+b}$, and consequently, for us, 2^0 has to be 1, and $2^{1/2} = \sqrt{2}$, and…”

or

> “In this class, let’s agree that $a \times 0 = 0$ for all numbers a, even for $a = 0$, and see where that takes us.”

We have trained our students to survive the school system well. The demands for a version of the concrete - “Do I need to know this?” “Will this be on the test?” “Just tell me what to memorize.” - are all completely appropriate and valid survival techniques for what we as a nation have created for our students to handle.

But there is room for some “wiggle room,” to give hints of flexibility and creativity, to let kids’ imagination and joy run free for a moment here and there, even if questions remain unanswered. (Can I invent a notion of a triangle-root? What if I invented division before I ever thought of multiplication? What rules of arithmetic might I have first come up with then? What if in geometry parallel lines, instead, always met like they do in perspective art? What rules should the number “infinity” follow?)

Innovation, creativity, and agility of the intellect rarely come from just forward, forward, forward, more, more, more, do, do, do. There’s something worthwhile in returning to and questioning the basics. After all, true business genius often comes from taking an old and simple idea and pushing it in a completely new direction. We can work to foster that thinking.

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