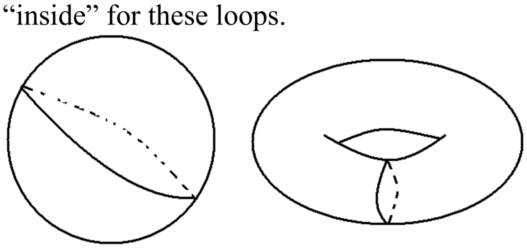
TANTON'S TAKE ON …

"AREA"

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

AUGUST 2012

It is very hard to pin down what exactly we mean by the area of a geometric figure. We might say it is "the amount of space inside the shape" which may be a fine definition of sorts for intuitive purposes, but it is not a very clear statement as a precise definition: What do we mean by "space" and what do we mean by "amount" of it? Even the word "inside" is problematic. Does every figure have a well-defined inside? For example, a loop drawn on the surface of a sphere divides the surface into two pieces – which is the inside? Worse, some loops drawn on the surface of a torus (donut) don't even divide the surface into two pieces! There is no hope of a meaningful definition of "inside" for these loops.

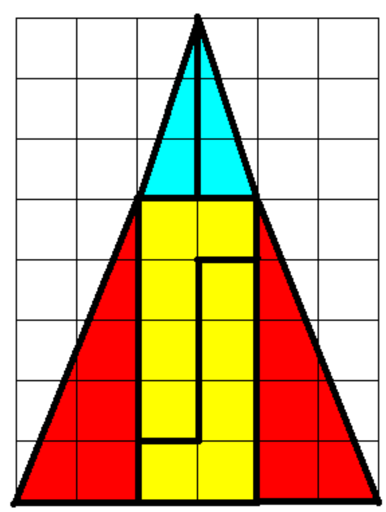


What is the "inside" for these loops?

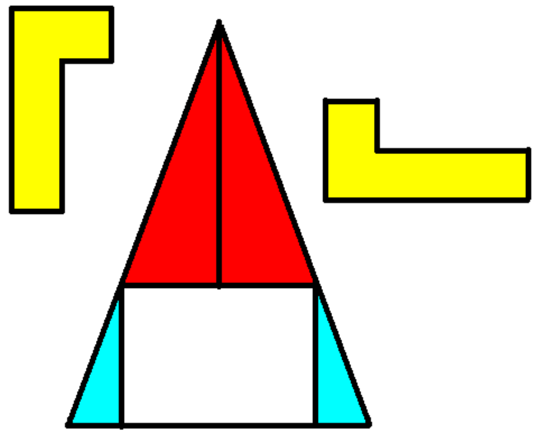
Maybe for planar figures we feel there is a well-defined notion of inside. Surely it is true that every loop drawn in the plane divides the plane into two pieces – one infinitely big and one not. (And let's call the small one the "inside.") Yet this issue was a serious concern for nineteenth-century mathematicians. With the discovery of bizarre fractal-like loops it was no longer clear that all loops in the plane divide two-dimensional space into two pieces.

Putting deep philosophical concerns aside, it is not clear that even our intuition about how "area," whatever it actually is, behaves. I like to start my unit on AREA in my geometry course for high-schoolers with the following classic activity.

Cut out an isosceles triangle, 8 units high and 6 units wide. Draw a horizontal line across the triangle 3 units down from the apex (the bottom of the blue triangles) and from there draw the remaining lines to divide the triangle into six pieces as shown. Draw the outline of the triangle on a separate clean sheet of paper.



Now pull those six pieces apart. Fit the two blue triangles in the left and right corners of the outline of the triangle and the two red triangles at the top as shown next.



Now fit the remaining two yellow pieces in the central space that remains.

ACTUALLY DO THIS!
YOU ARE IN FOR A SHOCK!

OR if you prefer to watch me do it go to the video http://www.jamestanton.com/?p=885.

Exercise: It is my birthday this month and I am going to bake a triangular cake. Show how I can cut and take out a piece of cake, divide the remainder of the cake into six pieces, rearrange them, and repeat! I am set for an infinite supply of cake! (Better yet , do this with gold leaf!)

I do this activity with my students. I have them cut out these triangles, rearrange pieces and be utterly surprised. Everything they thought was true about how area should behave is thrown out the window!

The discussion that follows invariably leads to two "facts" we like to believe about area, which students usually phrase as follows: *The area of a shape does not change if we move it* and *If we subdivide a shape into pieces, the area of the whole shape equals the sum of areas of the pieces.*

Yet somehow this triangle paradox seems to challenge the validity of these two self-evident truths!

COMMENT: Have you actually done the triangle trick? Have you watched me do it? Please do! The trick is actually an illusion – but you need to see it. All is revealed and explained in the remaining videos at http://www.jamestanton.com/?p=885 .

I discuss the illusion and have the kids explain it in my classes. Apart from being delightfully quirky and fun, this activity has brought up and pinned down two key postulates. (And since the students brought them up themselves, they <u>own</u> these ideas. The math is their creation!)

Rephrasing their ideas into "geometry speak" gives:

> Area Congruence Postulate (I):
> *Congruent shapes have equal areas.*
>
> Area Addition Postulate (II):
> *If a figure can be regarded as a union of two figures that overlap only along line segments, then the total area of the figure equals the sum of the areas of the two sub-figures.*

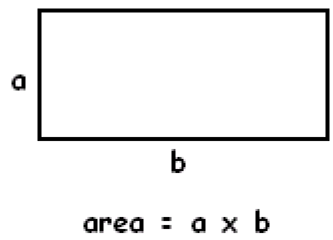
Of course, these postulates don't explain what "area" actually is, nor even give us a single example of what area could be for any one kind of shape! We need to rely on one more piece of intuition in order to develop a suitable theory of "area" for geometry.

As a starting point, it seems reasonable to say that a 1×1 square should have area one. We call this a basic unit of area. As four of these basic units fit snugly into a square with side-length two, overlapping only along line segments, postulate II suggests that a 2×2 square has area four. A 3×6 rectangle holds 18 basic unit squares and so has area 18. In general, a rectangle that is a units long and b units wide, with both a and b whole numbers, has area $a \times b$:

$$\text{area of a rectangle} = \text{length} \times \text{width}$$

This is a fundamental formula. To put the notion of area on a sound footing, we go the extra step and use this formula as a defining law!

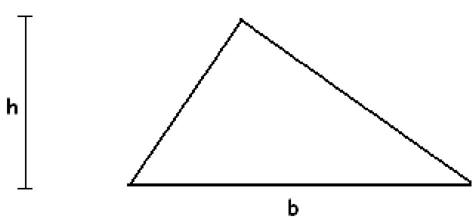
The area of *any* rectangle is defined to be the product of its length and its width.



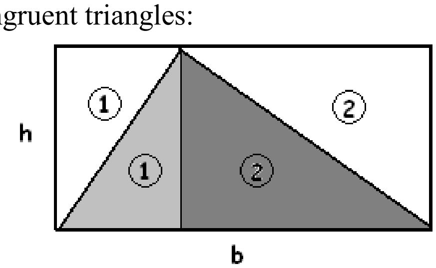
And we now mean *ANY* rectangle. Although it is not possible to fit a whole number of unit squares into a rectangle that is $5\frac{3}{4}$ units long and $\sqrt{7}$ units wide, for example, we declare, nonetheless, that the area of such a rectangle is the product of these two numbers. (This agrees with our intuitive idea that, with the aid of scissors, about $5\frac{3}{4}\times\sqrt{7}\approx 15.213$ unit squares will fit in this rectangle.)

Okay, with postulate I and postulate II and the declaration that the area of a rectangle is length times width, we are all set develop a theory of area. We can now follow the standard material in a high school geometry text book.

EXAMPLE: **AREA OF A TRIANGLE**
Consider a triangle of base length b and height h:



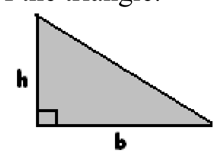
Enclose the triangle in a rectangle to see congruent triangles:



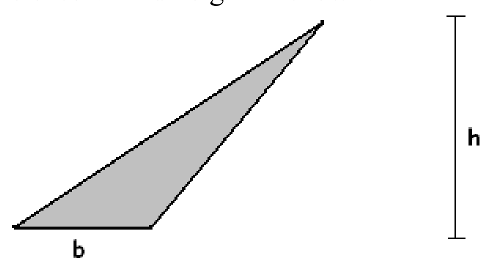
Areas labeled "1" are congruent and so have matching areas, as do the areas labeled "2". Also, the entire figure is composed of four small triangles that overlap only along their edges. By the two postulates, we are forced to conclude that the area of the triangle we seek, area 1 plus area 2, is half the area of the entire rectangle. So:

Area of triangle
$$=\frac{1}{2}\times \text{area of rectangle} = \frac{1}{2}bh$$

COMMENT: This formula also holds for a right triangle, with the altitude matching one of the sides of the triangle.

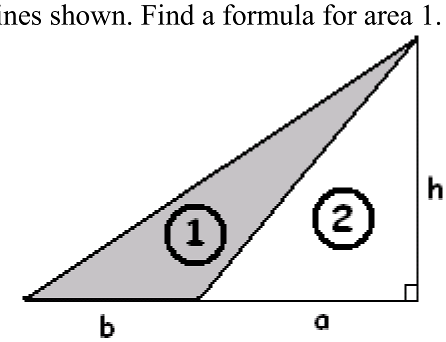


QUESTION: Does this formula hold for obtuse triangles, even if we insist on using the base b and height h shown?



COMMENT: This is a real question! I've had both students and teachers alike struggle over this issue. We've all been programmed to say "area is half base times height" but it is not at all obvious that that formula should work for this type of triangle. After all, enclosing it in a rectangle as we did before does not help! (Try it!)

Exercise: Draw and label the additional lines shown. Find a formula for area 1.



As every polygon can be subdivided into triangles we have now, in theory, a means to compute the areas of all polygons. Of course, the remainder of a standard geometry chapter is to derive formulas for the areas of special polygons: parallelograms, trapezoids, regular polygons, and so on. Have fun with all that!

And just when we think all is set and mighty good …

A TRULY DISTURBING PARADOX

We started with a fundamental shape, in our case a rectangle, and *asserted* it to have "area" given by a certain formula. From this a general theory of area for other geometric shapes followed. One can apply such an approach to develop a theory for measuring the size of other characteristics of objects - the "surface area" of three-dimensional solids, or their "volume." One can also develop a number of exotic applications.

Although our definition for the area of a rectangle is motivated by intuition, the formula we developed is, in some sense, arbitrary. Defining the area of a rectangle as given by a different formula could indeed yield a different, but consistent, theory of area.

In 1924 Stefan Banach and Alfred Tarski stunned the mathematical community by presenting a mathematically sound proof of the following assertion:

It is theoretically possible to cut a solid ball into nine pieces, and by reassembling them, without ever stretching or warping the pieces, form TWO solid balls each exactly the same size and shape as the original.

This result is known as the Banach-Tarski paradox and its statement—proven as a mathematical fact—is abhorrent to our understanding of how area and volume should behave: the volume of a finite quantity of material should not double after rearranging its pieces! That our intuitive understanding of area should eventually lead to such a perturbing result was considered very disturbing.

What mathematicians have come to realize is that "area" is not a well-defined concept: not every shape in a plane can be assigned an area (nor can every solid in three-dimensional space be assigned a volume). In any theory of area and volume, there exist certain nonmeasurable sets about which speaking of their area is meaningless. The nine pieces used in the Banach-Tarski paradox turn out to be such nonmeasurable sets, and so speaking of their volume is invalid. (They are extremely jagged shapes, fractal in nature, and impossible to physically cut out.) In particular, interpreting the final construct as "two balls of equal *volume*" is not allowed.

Our simple intuitive understanding of area works well in all practical applications. The material presented in a typical high-school and beginning college curriculum, for example, is sound. However, the Banach-Tarski paradox points out that extreme care must be taken when exploring the theoretical subtleties of area and volume in greater detail.

tanton.math@gmail.com