TANTON'S TAKE ON ...

LOGARITHMS

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

APRIL 2013

Try walking into an algebra II class and writing on the board, perhaps even in silence, the following:

$$power_2(8) = 3$$

$$power_5(25) = 2$$

Your turn!

$$power_3(27) = \_\_\_\_\_$$

$$power_{10}(100) = \_\_\_\_\_$$

$$power_4(16) = \_\_\_\_\_$$

$$power_4(64) = \_\_\_\_\_$$

$$power_7\left(\frac{1}{7}\right) = \_\_\_\_\_$$

$$power_2\left(\sqrt{2}\right) = \_\_\_\_\_$$

$$power_{\frac{1}{3}}(9) = \_\_\_\_\_$$

$$power_{10}(million) = \_\_\_\_\_$$

$$power_{73}(1) = \_\_\_\_\_$$

$$power_{0.01}(1000) = \_\_\_\_\_$$

$$power_{100}(0.1) = \_\_\_\_\_$$

$$power_{\sqrt{6}}\left(\frac{1}{36}\right) = \_\_\_\_\_$$

$$power_1(5) = \_\_\_\_\_$$

I bet your students, perhaps after some initial mumbling amongst themselves, will start to correctly fill in the blanks. Great! You have just taught logarithms!

Congratulate your students for their cleverness, and then say: "Unfortunately, for unusual historical reasons, we don't use the word *power*. We use the strange word *logarithm,* which we write as *log* for short." Now go though the list on the board and cross out each word *power* and replace it with *log*.

~~power~~ log $_3(27) = 3$

~~power~~ log $_{10}(100) = 2$

~~power~~ log $_4(16) = 2$

~~power~~ log $_4(64) = 3$

~~power~~ log $_7\left(\frac{1}{7}\right) = -1$

~~power~~ log $_2\left(\sqrt{2}\right) = 1/2$

~~power~~ log $_{\frac{1}{3}}(9) = -2$

~~power~~ log $_{10}(million) = 6$

~~power~~ log $_{73}(1) = 0$

~~power~~ log $_{0.01}(1000) = -3/2$

~~power~~ log $_{100}(0.1) = -1/2$

~~power~~ log $_{\sqrt{6}}\left(\frac{1}{36}\right) = -4$

~~power~~ log $_1(5) =$ huh?

Taking the time to do this in a showy way brings home the point that logarithms are just powers. "Whenever we see the word *log* we are to think *power*."

> $\log_b(x)$ is just $power_b(x)$, the power of b that gives the answer x.

Now finish the lesson by recounting the historical story detailed next!

THE HISTORICAL MOTIVATION for LOGARITHMS

During the Renaissance in Europe the study of science really took off. Scholars were doing experiments and collecting data left, right and center. With the invention of the telescope (Galileo first called the device a *perspicillum*) astronomers recorded the positions of heavenly objects and tracked their motions with higher and higher levels of precision. Geographers were using trigonometry and making measurements of land features on the Earth.

But advances in all this glorious work were severely held back by one very annoying issue: all calculations had to be done by hand (no calculators in the 1500s) and paper-and-pencil arithmetic is slow and tedious!

Adding data values (say, to calculate the average of ten experimental results) is not too onerous. (Could you compute $2.313 + 3.097 + 2.872$ fairly easily by hand?) But multiplying multi-decimal numbers, as one might need to do for scientific formulas, is mighty laborious! (Care to work out $2.313 \times 3.097 \times 2.872$ without a calculator?) It seemed totally absurd to have significant advances in science held back by the simple fact that performing multiplication by hand is so slow and laborious (and very much open to making errors!).

So in the late 1500s a Scottish mathematician by the name of John Napier (1550 – 1617) took it upon himself to help his beleaguered scientific colleagues. He set out to devise a method that would turn multiplication problems into addition problems.

He succeeded in this task, but his approach was highly creative to say the least!

To find a product $M \times N$ Napier envisioned two objects moving along a section of a straight line. He gave the first object a velocity related in some complicated way to the number M, and the second object a velocity which varied in a way related to the number N and to the distance the first car still needed to travel along the line. He found that in his strange method computing the ratio of the velocities of the two objects had, in essence, converted the multiplication problem $M \times N$ into a calculation of addition. (Napier also changed the scale of this problem so that the number $10^7 = 10{,}000{,}000$ would play a prominent role. He felt this would assist geographers and astronomers who were often dealing with large numbers.)

Napier based the name for his technique on the Greek words *logos* for ratio and *arithmos* for number to get *logarithm.* Napier's methods were tremendously successful and highly praised, despite being so complicated and hard to understand in the theory.

So that scholars would not have to worry about conceptual details Napier, with the help of Henry Briggs (1561-1630), published tables of logarithm values. This made the computation of $M \times N$ extraordinarily straightforward:

1. Look up the values $\log M$ and $\log N$ in the table.
2. Add the two numbers.
3. Look back at the table and see which entry has $\log M + \log N$ as its value. This entry is the product $M \times N$.

It wasn't until almost a century later that scholars realized that Napier's logarithms are actually very simple and already familiar: they are nothing more than exponents computed backwards! But, of course, by then the word *logarithm* was firmly in place and the name stays with us to this day. (With absolutely no disrespect to Napier, life would be so much easier for students today if we let go of the work *logarithm* and used the word *power*.)

Comment: Henry Briggs suggested to Napier he base his logarithms on the number 10. This, after all, is the number on which our arithmetic system. Today base-ten logarithms are sometimes called *Briggsian Logarithms*. They are also called *common logarithms* and the subscript ten is usually omitted in their use. (Thus, for instance, $\log 7$ is understood to mean $\log_{10} 7$.)

By the way… In today's notation, Napier's logarithms would be written: $10^7 \log_{1-\frac{1}{10^7}} \left(\frac{N}{10^7}\right)$. You can see now why it was so hard to recognize Napier's method as simple exponentiation in reverse!)

LOGARITHMS REALLY DO DO THE TRICK!

Let's establish first some basic properties of logarithms. Understanding their validity is simply a matter of pure thought!

Recall: $\log_b(x) = power_b(x)$!

PROPERTY 1: $\log_b(b) = 1$

Reason: The power of *b* that gives the answer *b* is obviously one!

PROPERTY 2: $\log_b(1) = 0$

Reason: The power of *b* that gives the answer 1 is 0.

PROPERTY 3: $\log_b(b^x) = x$

Reason: The power of *b* that gives the answer b^x is obviously… x !

PROPERTY 4: $b^{\log_b(x)} = x$

(This one throws people!)

Reason: Recall that $\log_b x$ is the power of *b* that gives the answer *x*. So if we use it as a power of *b*, we must obtain the answer … *x*!

COMMENT: Properties 3 and 4 show that exponentiation and taking logarithms "undo" each other. They are inverse operations (as mathematicians in the 1700s finally noticed!)

PROPERTY 5: NAPIER'S DREAM

$$\log_b(N \times M) = \log_b N + \log_b M \ .$$

Logarithms convert multiplication problems into addition problems.

EXAMPLE: Consider $\log_2(8 \times 32)$.

This, by definition, is the power of 2 that gives the answer 8×32 . Now, 8 is a product of three twos ($\log_2(8) = 3$) and 32 is the product of five twos ($\log_2(32) = 5$). Thus 8×32 is a product of $3 + 5$ twos. We have:

$$\log_2(8 \times 32) = 3 + 5 = \log_2(8) + \log_2(32)$$

This example offers a hint that logarithms do indeed accomplish what Napier set out to do. Although this example is helpful, we still need a formal proof of this property that works for *all* numbers, not just on convenient whole number exponents.

Formal Reason: We want to prove that $\log_b(M \times N)$ is $\log_b(M) + \log_b(N)$. That is, we want to show that $\log_b(M) + \log_b(N)$ is the right power of b to use to get the answer $M \times N$.

Let's see if it is.

$$b^{\log_b M + \log_b N} = b^{\log_b M} \cdot b^{\log_b N} = M \cdot N \ .$$

(Here we used the exponent rule $x^{a+b} = x^a \cdot x^b$, and then we used prop. 4.)

YES indeed! $\log_b(M) + \log_b(N)$ is the power of b that gives the answer $M \times N$!

EXERCISE: I tell you that, for some number *b*:

$$\log_b 2 = 0.693$$
$$\log_b 3 = 1.098$$
$$\log_b 5 = 1.609$$

Without a calculator, find $\log_b 4$, $\log_b 6$, $\log_b 8$, $\log_b 9$, $\log_b 10$ and $\log_b 600$.

Estimate $\log_b 7$ and $\log_b 70$.

Also estimate

$$\log_b\left(1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\right).$$

EXERCISE: Establish the rule

$$\log_b\left(\frac{N}{M}\right) = \log_b N - \log_b M$$

Comment: As division is the reverse process of addition, it is not surprising that subtraction appears here.

Although property 5 was the reason for the invention of logarithms in the year 1600, this need is no longer relevant for today. We multiply numbers with calculators!

However, there is a sixth property of significant importance to us still.

PROPERTY 6: $\log_b(M^x) = x\log_b(M)$.

EXAMPLE:

$$\begin{aligned}&\log_b(M^4)\\ &= \log_b\left(M\times M\times M\times M\right)\\ &= \log_b(M)+\log_b(M)+\log_b(M)+\log_b(M)\\ &= 4\log_b(M)\end{aligned}$$

Of course, this approach relies on the convenience of whole number exponents.

Formal Reason: Property 6 claims that the power of b we need to get the answer M^x is $x\log_b(M)$. Let's check! (We'll again use exponent rules and property 4.)

$$b^{x\log_b M} = \left(b^{\log_b M}\right)^x = M^x.$$

YES! $x\log_b(M)$ is indeed $power_b\left(M^x\right)$

EXERCISE: Which numbers are suitable bases for logarithms?

Does $\log_1(M)$ ever make sense?

Does $\log_0(M)$ ever make sense?

$\log_{-2}(4) = 2$. Does $\log_{-2}(M)$ always make sense?

Does $\log_{\sqrt{3}}(M)$ always make sense?

Does $\log_{0.0000003}(M)$ always make sense?

Does $\log_{\frac{13}{19}}(M)$ always make sense?

EXAMPLE: Solve $3^x\cdot 4^{x+2} = 7\cdot 5^{3x-2}$

Answer: Property number 6 shows how to bring exponents down from an expression. Let's apply a logarithm to both side of the equation. We can use any base of our choice. Let's use base-10 logarithms (as they appear on our calculators!).

We have:

$$\log\left(3^x\cdot 4^{x+2}\right) = \log\left(7\cdot 5^{3x-2}\right)$$
$$\log\left(3^x\right)+\log\left(4^{x+2}\right) = \log 7 + \log\left(5^{3x-2}\right)$$
$$x\log 3 + (x+2)\log 4 = \log 7 + (3x-2)\log 5$$
$$x\log 3 + x\log 4 - 3x\log 5 = \log 7 - 2\log 5 - 2\log 4$$

yielding

$$\begin{aligned}x &= \frac{\log 7 - 2\log 5 - 2\log 4}{\log 3 + \log 4 - 3\log 5}\\ &= \frac{\log 7 - \log 25 - \log 16}{\log 3 + \log 4 - \log 125}\\ &= \frac{\log(7/400)}{\log(12/125)}\end{aligned}$$

EXAMPLE: How do I work out $\log_5(7)$ on my calculator?

Answer: Let's play with it. If $x = \log_5 7$, then x is the power of 5 that gives the answer 7:

$$5^x = 7.$$

We have now an expression with no mention of logarithms. Let's apply a logarithm of our choice, the common logarithm, to each side of the equation.

$$x \log 5 = \log 7$$

This gives:

$$x = \frac{\log 7}{\log 5}.$$

I can put this into my calculator!

Comment: Please don't memorise a "change of base" formula.
Like all things in math, JUST DO IT!

INTERNET RESEARCH: Find the details of Napier's "ratio of velocities."

VIDEOS: Here are some short videos recapping the history and some of the mathematics of logarithms. Enjoy!

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