TANTON'S TAKE ON …

GRAPHING QUADRATICS

CURRICULUM TIDBITS FOR THE MATHEMATICS CLASSROOM

APRIL 2012

Welcome to a new Mathematics Letter!

Dr. James Tanton is new to the D.C. area, but for the past eight years has been the founding director of the St. Mark's Institute of Mathematics up in the Boston Area.

He is a research mathematician (PhD Princeton, 1994) deeply interested in uniting the mathematics experienced by school students and the creative mathematics practiced and explored by mathematicians. He has worked as a full-time high school teacher and does all that he can to bring joy into mathematics learning and teaching.

James writes math books. He gives math talks and conducts math workshops. He teaches students and he teaches teachers. He publishes articles and papers, always thinking, creating and doing new math. And he shares the mathematical experience with students of all ages, helping them publish research papers too!

Welcome to his first curriculum letter from Washington D.C.!

The Algebra II Curriculum:
The detailed study of quadratic equations – their algebra and their graphs – constitutes a significant component of the algebra II curriculum. There students are usually taught to recognize equations of the form

$$y = ax^2 + bx + c$$

as having U-shaped graphs, with a vertex at a location on the x-axis given by a memorized formula $x = -b/2a$, as crossing the x-axis twice if the "discrimminant" $b^2 - 4ac$ is positive, and being upward facing or downward facing depending on the sign of the coefficient a.

This is all fine and good, but it is so easy for the student experience in all this to fall into a joyless "memorise and do" enterprise.

Have you ever asked …? The prefix *quad-* means "four." But quadratic expressions are ones that involve powers of x up to the second power (not the fourth power). So why are quadratic equations associated with the number four? Shouldn't we have a name that is about the number two: Diatics? Duo-atics? Bi-atics? Maybe Two-datics? Equations with x^3 are *cubics*, those with x^4 *quartics*. What's up with quadratics?

When I teach quadratics we follow a general idea that we should ask about any equation: **Are there any x-values that stand out as interesting?**

For example, in $y=(x-3)^2+8$ the value $x=3$ stands out as interesting.

In $y=(x-5)(x+13)+20$, the values $x=5$ and $x=-13$ are interesting: they both produce the output $y=20$.

This is an approach that pervades so much of mathematics, so let's play with this theme in algebra II. Here's how this idea makes the graphing of quadratics a piece of cake – with no memorization of formulas needed!

COMMENT: As a teacher I believe it is important to teach kids a sense of self-reliance, the ability to "nut their way" through a problem (what a great life skill!) and not to rely on memorized work that brings only short-term success. I do not promote the memorization of any formula!

One prerequisite … In my next curriculum letter I'll show how the algebra of quadratics makes it clear that every graph of the form $y=ax^2+bx+c$ is a <u>symmetrical</u> U-shaped graph.

This is not at all obvious! It is clear that $y=x^2$ gives a symmetrical U-graph. But what if we "add" to it the diagonal line $y=x$? Should $y=x^2+x$ be perfectly symmetrical? Should it even be U-shaped? Have a look at: http://www.jamestanton.com/?p=574 if you would like to see a video addressing this issue right now.

Here's how to graph some quadratics once we're confident they are all going to be symmetrical Us.

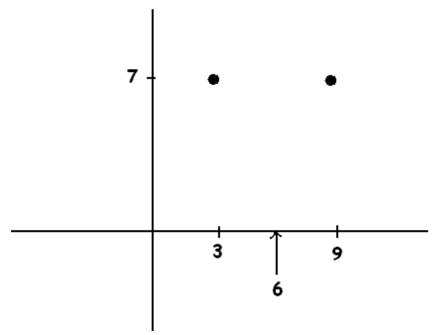
EXAMPLE:
Sketch $y=2(x-3)(x-9)+7$

Answer: The first thing to note is that if we were to expand $2(x-3)(x-9)+7$ we would obtain an expression of the form: ax^2+bx+c. It is going to be a symmetrical U.

Two obvious and interesting x-values stand out:

Put $x=3$ and we get $y=0+7=7$.
Put $x=9$ and we get $y=0+7=7$.

So we have U that goes through two symmetrical points:

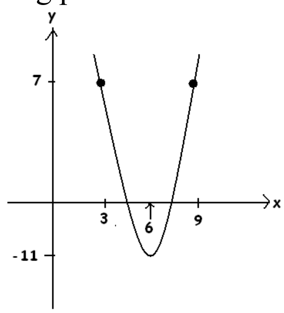


Since the graph is symmetrical, common sense says that the graph must have vertex at $x=6$, the point halfway between $x=3$ and $x=9$.

The trouble is we don't know how high the U is at this point: Does the U sit above the x-axis? Dip below it? Just touch it? Is the U right-side up or upside down?

Well, let's plug $x=6$ into the formula and find out! When $x=6$ we have $y=2(3)(-3)+7=-11$

Common sense says that we must have the following picture:



> **EXERCISE:** Use this technique to sketch $y = -(x+2)(x-10)+1$.
> Where is its vertex? What is its y-intercept? What is the graph's largest value? Just use a picture and common sense to answer all these questions!

How does one find interesting x-values if they are not obvious?

Let's address this with an example. Consider $y = x^2 + 4x + 5$.

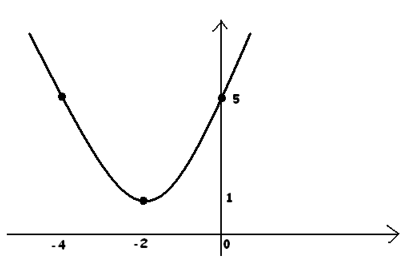
To find interesting x-values, focus on the "x part" of the formula: $x^2 + 4x$.

This can be rewritten as : $x(x+4)$ and so:

$$y = x(x+4)+5$$

Now it is clear that $x = 0$ and $x = -4$ are interesting. They both give $y = 5$. By symmetry, we know the vertex lies midway between these interesting x-values, at $x = -2$, and here $y = (-2)(2)+5 = 1$.

Here then is its graph!



> **EXAMPLE:** Find a formula for the following U-shaped curve:
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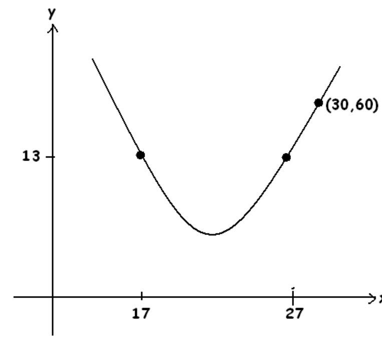
> 
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Answer: The picture shows us that $x = 17$ and $x = 27$ give symmetrical outputs. This suggests the formula:

$$y = a(x-17)(x-27)+13$$

We inserted a value a as there could be a number out front.

However, we are also told that when $x = 30$ we have $y = 60$. So we do need:

$$60 = a(13)(3)+13$$
$$47 = 39a$$
$$a = \frac{47}{39}$$

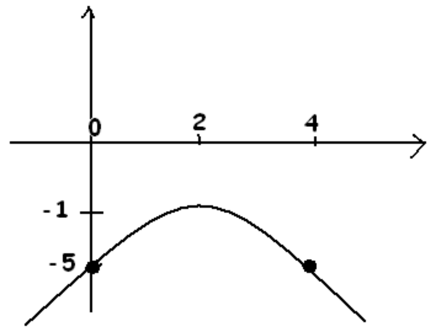
So $y = \frac{47}{39}(x-17)(x-27)+13$ does the trick! □

Who needs the discriminant?

> **EXAMPLE:** How many times does $y = -x^2 + 4x - 5$ cross the x-axis?

Answer: We have $y = -x(x-4)-5$ and so $x = 0$ and $x = 4$ are interesting. (They both give $y = -5$.)

The vertex thus occurs at $x = 2$ and here $y = -4 + 8 - 5 = -1$.



We see this graph crosses the x-axis no times. Done via pure understanding and common sense!

PRACTICE:

a) Find a value for k so that $y = 5x^2 - 10x + k$ just touches the x-axis.

b) Find a value for m so that $y = -2x^2 - 18x + m$ has largest value 100.

c) Find a value for a so that $y = (x-a)(x-3a)$ has smallest value -10.

HINT: Use symmetry to draw a graph as best you can in each case. You can nut your way through each of these!

FOR THOSE WHO WANT A FORMULA …

Where does the vertex of $y = ax^2 + bx + c$ lie?

Well, in our hunt for interesting x-values we would focus on the x-part of the equation and write:

$$y = x(ax + b) + c$$

We see that both $x = 0$ and $x = -\frac{b}{a}$ give the same output of $y = c$. By symmetry, the vertex must lie half-way between these two x-values, namely at:

$$x = -\frac{b}{2a}.$$

I personally find it too hard to keep this in my mind. If someone hands me a quadratic I'd rather do what I do with most any equation and ask:

Does it have symmetry? If so, can I make use of interesting inputs to exploit that symmetry?

In algebra II we are providing students a first means to develop such systematic self-reliant thinking. The study of quadratics really could be the study of exploiting symmetry.

A COMMENT: This graphing technique appears as a video at http://www.jamestanton.com/?p=370 . You will find more videos on the issue of algebra II quadratics at the general website under THINK CURRICULUM!

ANOTHER COMMENT: We haven't discussed here how to handle "repeated" interesting x-values. For instance, do we have the means of "nutting our way towards" the graph of $y = (x-3)^2 + 5$? The answer is YES! Care to think about how to help students with this?

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