TANTON'S TAKE ON …

EXPECTED VALUE

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This essay is inspired by a puzzle I posted on twitter.

If I roll a die five times in a row, how many distinct values do I expect to see?

For example, in rolling 3, 5, 3, 6, 5 I would count three distinct values, and in rolling all 6s I would count one distinct value.

One can, of course, answer this puzzle by listing out all $6^5 = 7776$ possible outcomes and determine the average count of distinct values among them. In fact, such a technique, if conducted only in theory, explains what we mean by the *expected value* of a *random variable* in a probabilistic scenario.

Suppose we are looking at an activity that relies on chance. By a *random variable* we simply mean a quantity whose value we don't yet know, but will as soon as we run the activity. (And running the activity a second time will likely produce a different value for that quantity, a third run of the activity yet another value, and so on.)

In our example, let's call it example 1, let X be the count of distinct die values we see in rolling a die five times in a row. For the rolls 3,5,3,6,5 we get $X = 3$, for the rolls 6,6,6,6,6 we get $X = 1$.

Example 2: I pay $1 to play a simple coin-toss gambling game. If it lands heads I win $3, if it lands tails I win nothing. Let Y be the random variable representing my profit in play a round of the game. Then $Y = 2$ for any round which results in heads, $Y = -1$ otherwise.

Example 3: In another gambling game I roll a die. If it lands 5 or 6, I win $10; if it lands 4 I win $5; otherwise I must pay $3. Let Z be the random variable representing my profit in a play of a round of the game. Then Z will have one of the values 10, 5, or -3.

Suppose we play the game of Example 2 a large number of times, say 600 times. Then, we'd expect, in ideal thinking, that 300 of the coin tosses will be heads and 300 will be tails. That is, we would win $2 three-hundred times and "win" -$1 three-hundred times. Our average profit is

$$\frac{2\times 300+(-1)\times 300}{600} = 2\times\frac{1}{2}+(-1)\times\frac{1}{2} = \frac{1}{2}.$$

We win, on average, 50 cents per game. That is, the average value we assign to Y is 0.5.

Suppose we play the game of example 3 also 600 times. Then, in ideal thinking, 200 times we would win $10, 100 times win $5, and lose $3 300 times. Our average profit will be

$$\frac{10\times 200+5\times 100+(-3)\times 300}{600} = 10\times\frac{2}{6}+5\times\frac{1}{6}+(-3)\times\frac{3}{6} = 2\frac{2}{3}.$$

We win, on average, two and two-thirds dollars per game. The average value we assign to Z is thus $2\frac{2}{3}$.

The average value of random variable W one expects to see in repeating the activity a large number of times is called the *expected value* of the random variable. It is denoted $E(W)$.

So we have $E(Y) = 0.5$ and $E(Z) = 2\frac{2}{3}$, and example 1 is asking for the value of $E(X)$.

Our calculations of $E(Y)$ and $E(Z)$ reveal a general formula for an expected value.

Suppose a random variable W for an activity can take the numerical values w_1, w_2, w_3, … and the probability that it takes value w_i is p_i. In repeating the activity a large number of times, say 600 times, the number of times we'd expect to see the value w_1 is $p_1\times 600$ times, the number of times we'd expect to see value w_2 is $p_2\times 600$ times, and so on. Thus, the average value of W we expect to see is

$$E(W) = \frac{w_1\times 600 p_1 + w_2\times 600 p_2 + \cdots}{600} = w_1\times p_1 + w_2\times p_2 + \cdots.$$

In rolling a die five times in a row, the probability of seeing one distinct value is $p_1 = \frac{6}{7776} = \frac{1}{1296}$, so

$$E(X) = 1\times\frac{1}{1296} + 2\times p_2 + 3\times p_3 + 4\times p_4 + 5\times p_5$$

where p_i is the probability of seeing i distinct values. (These numbers are hard to compute!)

SOME THEORETICAL PROPERTIES OF EXPECTED VALUE

If we double all the payouts (and fees) of a gambling game, then we would expect the average profit or loss that a player experiences doubles as well.

Result: $E\left(2W\right) = 2 \times E\left(W\right)$.

Proof: If W is a random variable that takes on numerical values w_1, w_2, w_3, … with respective probabilities p_1, p_2, p_3, …, then by "$2W$" we mean the random variable that takes on the values $2w_1$, $2w_2$, $2w_3$, … with the same respective probabilities. We have

$$\begin{aligned} E\left(2W\right) &= 2w_1 \times p_1 + 2w_2 \times p_2 + \cdots \\ &= 2\left(w_1 \times p_1 + w_2 \times p_2 + \cdots\right) \\ &= 2E\left(W\right). \end{aligned}$$

It is sometimes meaningful to add together the values of two different random variables. For example, if I play the gambling games of examples 2 and 3 simultaneously (that is, I toss a coin and roll a die at the same time), then $Y + Z$ represents my sum profit or loss. (For example, if I toss a tail and roll a 4, then $Y + Z$ has the value $\left(-1\right) + 5 = 4$.) It feels right, in this scenario at least with two games whose outcomes in no way influence each other, that the expected value of a sum of two random variables should equal the sum of their individual expected values. This result is true in general.

Result: $E\left(W + V\right) = E\left(W\right) + E\left(V\right)$.

Proof: Suppose W is a random variable that takes on numerical values w_1, w_2, w_3, … with respective probabilities p_1, p_2, p_3, …, and V is a random variable that takes on numerical values v_1, v_2, v_3, … with respective probabilities q_1, q_2, q_3, …. Then $W + V$ takes on the values $w_i + q_j$ for different values of i and j, but the probability of this specific sum occurring is unclear. Let's write

$$P_{i\,j} = P(W = w_i \text{ and } V = v_j)$$

for this probability.

Comment: If one believes the activities the two random variables represent are "independent," then one might conclude that $P(W = w_i \text{ and } V = v_j) = p_i \times q_j$. We do not need this to be the case in what follows.

Now

$$\begin{aligned} &E\left(W + V\right) = \\ &\left(w_1 + v_1\right) \times P_{11} + \left(w_1 + v_2\right) \times P_{12} + \left(w_1 + v_3\right) \times P_{13} + \cdots \\ &+ \left(w_2 + v_1\right) \times P_{21} + \left(w_2 + v_2\right) \times P_{22} + \left(w_2 + v_3\right) \times P_{23} + \cdots \\ &+ \left(w_3 + v_1\right) \times P_{21} + \left(w_3 + v_2\right) \times P_{32} + \left(w_3 + v_3\right) \times P_{33} + \cdots \\ &+ \cdots. \end{aligned}$$

This equals

$$\begin{aligned} &w_1 P_{11} + w_1 P_{12} + w_1 P_{13} + \cdots \\ &+ w_2 P_{21} + w_2 P_{22} + w_2 P_{23} + \cdots \\ &+ \cdots \\ +& \\ &v_1 P_{11} + v_2 P_{12} + v_3 P_{13} + \cdots \\ &+ v_2 P_{21} + v_2 P_{22} + v_3 P_{23} + \cdots \\ &+ \cdots \end{aligned}$$

which is

$$
\begin{aligned}
&w_1\left(P_{11}+P_{12}+P_{13}+\cdots\right)\\
&+w_2\left(P_{21}+P_{22}+P_{23}+\cdots\right)\\
&+\cdots\\
+&\\
&v_1\left(P_{11}+P_{21}+P_{31}+\cdots\right)\\
&+v_2\left(P_{12}+P_{22}+P_{32}+\cdots\right)\\
&+\cdots
\end{aligned}
$$

Now $P_{11}+P_{12}+P_{13}+\cdots$ is

$$
\begin{aligned}
&P(W=w_1 \text{ and } V=v_1)\\
&+P(W=w_1 \text{ and } V=v_2)\\
&+P(W=w_1 \text{ and } V=v_3)\\
&+\cdots
\end{aligned}
$$

with the values for V running through all possibilities. So for something in this list of options to actually occur all we need is $W=w_1$ at some time. That is, this sum of probabilities must equal $P\left(W=w_1\right)$, which is p_1.

Similarly, $P_{21}+P_{22}+P_{23}+\cdots$ is p_2 and $P_{15}+P_{25}+P_{35}+\cdots$ is q_5.

Thus $E\left(W+V\right)$ equals

$$
\begin{aligned}
&w_1p_1+w_2p_2+w_3p_3+\cdots\\
+&\\
&v_1q_1+v_2q_2+v_3q_3+\cdots
\end{aligned}
$$

which is indeed $E\left(W\right)+E\left(V\right)$.

ANSWERING THE PUZZLE

The key is to identify a different set of random variables for the roll of die five times that, when combined, count the number of distinct values that occur.

(My thanks to David Radcliffe for alerting me to this particular approach.)

Let X_1 be the random variable that takes the value 1 if a 1 appears some time among the five rolls, and the value 0 otherwise.

Let X_2 be the random variable that takes the value 1 if a 2 appears some time among the five rolls, and the value 0 otherwise.

Define X_3, X_4, X_5, and X_6 similarly.

Then count of distinct values among five rolls is given by

$$X=X_1+X_2+X_3+X_4+X_5+X_6.$$

So

$$
\begin{aligned}
E\left(X\right)&=E\left(X_1+X_2+\cdots+X_6\right)\\
&=E\left(X_1\right)+E\left(X_2\right)+\cdots+E\left(X_6\right)
\end{aligned}
$$

What is the value of $E\left(X_1\right)$?

Now the chances of seeing no 1s among five rolls is $\left(\frac{5}{6}\right)^5$, and so the chances of seeing at least one 1 is $1-\left(\frac{5}{6}\right)^5$.

Thus

$$E(X_1) = 1 \times \left(1 - \left(\frac{5}{6}\right)^5\right) + 0 \times \left(\frac{5}{6}\right)^5$$
$$= 1 - \left(\frac{5}{6}\right)^5.$$

The calculations for $E(X_2)$, …, $E(X_6)$ are identical.

Thus we have

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_6)$$
$$= 6\left(1 - \left(\frac{5}{6}\right)^5\right)$$
$$\approx 3.589.$$

Thus, if we roll a die five times in a row each day for a year and record the number of distinct values we see each day, then the average value of those counts will be mighty close to 3.589.

tanton.math@gmail.com