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TANTON'S TAKE ON …

EXTRANEOUS SOLUTIONS

SEPTEMBER 2016

Why is it that when performing a string of manipulations to solve an algebraic equation one should sometimes – but not always – go back and check that the solutions one obtains really are valid solutions to the original equation? What are extraneous solutions and why might they occur?

I was recently asked to write about this topic as it really does seem to be a point of confusion for many students. And rightly so! Some textbooks and curricula are hazy on this matter (if they discuss it at all).

MATHEMATICAL SENTENCES

As I wrote in the May 2016 Curriculum Essay "Math is a Language," http://tinyurl.com/zzz33gv, statements in mathematics really are English sentences.

For example, the statement "$2+3=5$" has a noun (the quantity two-plus-three), a verb (equals), and an object (five). The statement

$$\frac{3}{4}+\frac{5}{2}=\frac{3}{4}+\frac{10}{4}=\frac{13}{4}=3.25$$

is the long sentence "Three-quarters plus five-halves is equivalent to three-quarters plus ten-quarters, which equals thirteen-quarters, which equals 3.25."

Some mathematical sentences are true ($2+3=5$, for instance) and some are false ($5<1$, for example). We are usually interested in mathematical statements that are true.

CONTEXT AND SOLUTION SETS

One needs clear context in order to determine whether or not a particular statement is true. For example, we cannot determine if the sentence "Harold is more than six feet tall" is true without knowing which particular Harold of the world we are referring to.

But we could bring together all the tall Harolds from across the globe and say "Here's the group of people who each make the sentence true."

The equivalent of the name "Harold" in algebra class is the symbol "x." Usually x represents a number, but we don't know which particular number we are actually referring to. The mathematical statement "$x > 6$" has no context in and of itself and so we don't actually know whether or not it is a true statement. (Some values for x make this a true statement, others do not.)

Here is another statement without context.

$$x^2 + 1 = 10 \ .$$

Because it has no context it is neither true nor false. But like we did with the Harolds of the world, we could assemble all the numbers x could be that would make this a true sentence. In this example, that set is $\{-3, 3\}$.

> The *solution set* to an equation is the set of all the numbers the variable(s) in the equation could adopt to make the equation a true sentence about numbers. Any particular value in the solution set is called a *solution* to the equation.

The solution set to the statement $x > 6$ is all real numbers larger than six. The solution set to the statement $x = 5$ is the set $\{5\}$. (Make sure to fully understand the subtlety of this example!)

FORWARD – AND BACKWARD - THINKING

We have certain beliefs about equalities and inequalities. For example, if $A = B$, then we like to believe it follows that $2A = 2B$. This allows us to reason, for example, that if x has a value that makes the statement

$$\frac{x^2}{2} = 8$$

a true sentence about numbers, then that same value for x also makes

$$x^2 = 16$$

a true statement. As there are only two values that make $x^2 = 16$ true, namely 4 and -4, then the possible values of x we are considering must be one of these two values. (And I can see that, indeed, both -4 and 4 make $x^2 / 2 = 8$ a true statement about numbers.)

Notice my pedantic reasoning and writing:

> If x is a value that makes $\frac{x^2}{2} = 8$ a true statement, then x is also a value that makes $x^2 = 16$ a true statement. Consequently, x could only be either -4 or 4. One checks that both of these values actually make the original statement a true one.

This is very different from what one usually writes in algebra class:

$$\frac{x^2}{2} = 8$$

$$x^2 = 16$$

$$x = 4 \ or \ -4.$$

A significant amount of additional content is left unstated in these three lines.

The first and second lines imply that $\frac{x^2}{2} = 8$ and $x^2 = 16$ are "equivalent" statements.

Two algebraic statements are *equivalent* if the solution sets for the two statements are identical.*

Is it obvious that $\frac{x^2}{2}=8$ and $x^2=16$ are equivalent statements?

We reasoned earlier only that any solution to $\frac{x^2}{2}=8$ must also be a solution to $x^2=16$. This came from the belief that if $A=B$ is true, then $2A=2B$ must be true as well.

But we also believe that if $C=D$ is true, then $\frac{1}{2}C=\frac{1}{2}D$ also follows as true. Thus we can conclude that any solution to $x^2=16$ must also be a solution to $\frac{x^2}{2}=8$.

So the two equations do have identical solution sets and so are equivalent algebraic statements.

Also, the final two statements, " $x^2=16$ " and " $x=4\ \ or\ -4$ " are also equivalent.

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*Caveat: We are assuming here that the types of values deemed permissible for the variable(s) in each equation are understood and are considered the same. For example, the equations $\left\lfloor \frac{x}{2} \right\rfloor = \left\lceil \frac{x}{2} \right\rceil$ and $\lfloor x \rfloor = 2\left\lfloor \frac{x}{2} \right\rfloor$ have the same solution sets if we assume x must be an integer, but they have different solution sets if we assume x can be any real number.
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So the lines we write in algebra class are actually loaded with context. In our example,

$$\frac{x^2}{2}=8$$
$$x^2=16$$
$$x=4\ or\ -4.$$

actually reads "The statement $\frac{x^2}{2}=8$ has exactly the same solution set as the statement $x^2=16$, which has the same solution set as $x=4\ \ or\ -4$."

What we write in algebra class states BOTH the forward and backward thinking, though we logically only need the forward thinking to solve the algebraic problem.

BEGINNING ALGEBRA CLASS

The algebraic manipulations one conducts in a beginning algebra class tend to all be "reversible."

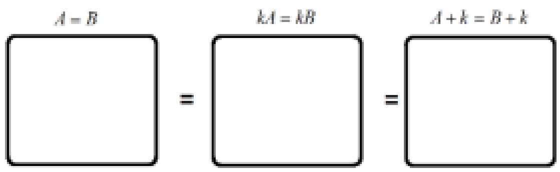
If $A=B$ holds, then $kA=kB$ must hold as well for any non-zero k. And if $kA=kB$ holds, then $A=B$ follows (by using the same belief with the constant $\frac{1}{k}$).

If $A=B$ holds, then $A+k=B+k$ must hold as well. And if $A+k=B+k$ holds, then $A=B$ follows (by using the same belied with the constant $-k$.)

Thus without thinking we implicitly (and correctly) assume that a whole string of statements like these

$$A=B$$
$$A+2=B+2$$
$$3A+6=3(B+2)$$
$$3A+1=3(B+2)-5$$

all have exactly the same solution sets. Our job then is to simply use these standard manipulations of algebra to obtain a statement whose solution set is blatantly obvious, one such as "$x = 5$" or "$x = 4$ *or* -4."

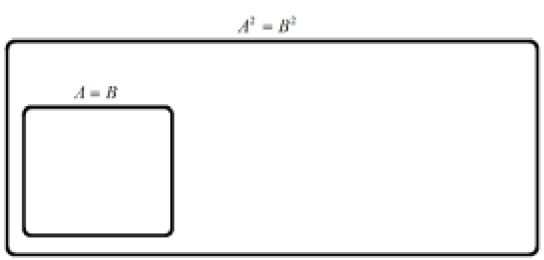


These equations always have the same solution sets.

This work forgets the forward-thinking care we first developed. And this should be kept in mind, because …

NOT ALL MANIPULATIONS PRESERVE SOLUTION SETS

Suppose x is value that makes the statement $x = 5$ a true statement. Then x also makes the statement $x^2 = 25$ true. This is because we believe that if $A = B$ holds, then $A^2 = B^2$ holds as well. But the converse need not be true: If x is a value that makes $x^2 = 25$ a true statement, it need not be the case that x also makes $x = 5$ are true statement.



Any solution to an equation $A = B$, must also be a solution to $A^2 = B^2$. That is all we know for sure.

Because careful algebraic thinking actually only requires only forward reasoning, we are permitted to use the squaring operation in our algebraic work.

Example: *Find the solution set to the equation* $x + 2\sqrt{x} = 24$.

Answer: If x is a value that makes the statement $x + 2\sqrt{x} = 24$ true, then it also makes

$$x + 2\sqrt{x} + 1 = 25$$

true. This can be rewritten

$$\left(\sqrt{x} + 1\right)^2 = 25.$$

Thus any solution that makes the original statement true must make

$$\sqrt{x} + 1 = 5 \ \ or \ \ -5$$

a true statement too. And such a value also makes

$$\sqrt{x} = 4 \ or \ \ -6$$

true. And if x is a value that makes this statement true, then it follows, by squaring, that

$$x = 16 \ \ or \ \ 36$$

will be true as well.

Thus the solutions to $x + 2\sqrt{x} = 24$ must be among the numbers $\{16, 36\}$.

Now, checking, we see that 16 makes the original equation a true statement about numbers, and 36 does not.

We are not surprised by the appearance of an "extraneous solution" since we performed the operation of squaring and so went to a possibly larger solution set.

There are many operations that might increase the size of the solution set of an equation $A = B$: squaring or raising to any even power, taking absolute value, multiplying each side by zero, for example. (What is the solution set to the equation $0 = 0$?)

One is welcome to perform any of these operations while manipulating an algebraic statement. Since we are only increasing the size of the possible solution set, we will conclude that any solution to the original statement will be among the elements of the final solution set we identify. Our job is to then check which of those, if any, are solutions to the initial equation.

Example: *Solve* $\sqrt{x}+2=0$.

SUBTLE DOMAIN ISSUES

An algebraic statement might, in and of itself, imply that there is a restriction on the permissible values of the variable. For example, in writing $\frac{1}{x}=\frac{x+3}{5}$, it is implied that x cannot be zero (otherwise we are dividing by zero). In writing $\sqrt{x-1}<4$, it is implied that x will adopt a value one or greater (otherwise we are attempting to take the square root of a negative quantity).

One should take note of an implied restrictions an algebraic equation might have.

Example: Solve $\frac{1}{x^2-4}=\frac{1}{x^2-5x+6}$.

Answer: It is implied here that x cannot be a value that gives us a denominator of zero in either fractional expression. Keeping that in mind, if x solves $\frac{1}{x^2-4}=\frac{1}{x^2-5x+6}$, then it also solves

$$\frac{x^2-5x+6}{x^2-4}=1,$$

and hence also solves

$$x^2-5x+6=x^2-4,$$

and

$$5x=10$$

and

$$x=2.$$

This final equation has solution set $\{2\}$. But 2 is not a solution to the initial equation (it is not even a permissible value). So the original equation has no solutions.

Example: Solve $\sqrt{x-3}+\sqrt{1-x}=0$.

Answer: Here it is implied that x adopts a value greater than or equal to 3 and also a value less than or equal to 1. Hmm. There are no solutions.

If we don't notice this, we could argue as follows: If x solves the original equation, then it also solves

$$\sqrt{x-3}=-\sqrt{1-x}$$

and hence also

$$x-3=1-x$$

(by squaring), and so solves $x=2$. The solution set of the original equation is a subset of $\{2\}$. Checking, we see it is the empty subset.

Example: Solve $|x-7|>2x-2$.

Answer: Recall that $|a|$ is a non-negative number and so is either a or $-a$, depending on whether or not a itself is non-negative or negative.

So let's look at two cases.

Case $|x-7|=x-7$.

Here our inequality reads $x-7>2x-2$. Any solution to this inequality is also a solution to $x<-5$.

But we have a problem! In this scenario we have subtly assumed that $x-7$ is a non-negative quantity, and so x is larger than or equal to seven. There are no solutions to $x<-5$ in this case.

Case $|x-7|=-(x-7)$.

Here our inequality reads $-x+7>2x-2$. Any solution to this is also a solution to $x<3$.

Here we have assumed that $x-7$ is a negative quantity, that is, that x is less than 7. The inequality $x<3$ has solution set "all numbers less than three" in this scenario.

So the solution set to the original inequality is a subset of all real numbers less than three. And one then reasons that this is indeed the solution set of the original equation.

FINAL PROBLEM: SKIPPING SOLUTIONS

A different issue can arise when attempting to multiply an equation through by a fractional quantity $\frac{1}{k}$ if the value of k is not actually known.

Consider the equation

$$x(x+2)=4(x+2).$$

The impulse is to cross out the common $x+2$ and just write $x=4$, and be done with it. But this skips a solution.

Here is the reasoning one should employ.

If $A=B$ holds, then $\frac{1}{k}A=\frac{1}{k}B$ holds too, for any non-zero value of k.

So if $x+2$ is non-zero, then we conclude that if x is a solution to $x(x+2)=4(x+2)$, then it is also a solution to $x=4$.

What if $x+2$ is zero?

Well, if $x+2=0$, then we see that $x(x+2)=4(x+2)$ reads as $0=0$, which is a true statement!

So the values of x that make $x(x+2)=4(x+2)$ true either make $x=4$ true or make $x+2=0$ true. The solution set must be a subset of $\{-2,4\}$, and we check that, indeed, both these values solve the original equation.

HOW MUCH WRITING?

Writing – and reading – paragraphs of text in solving algebra problems really does become quite tedious. One does want to just write a list of equations, one after the next.

But it should be understood by one and all that only "forward reasoning" is being assumed here, that any manipulation that involves more than adjusting by constants might increase the size of the solution set. Thus any algebraic solution should end with a final statement as to which elements, if any, of the final solution set actually are solutions to the original equation.

$$\frac{\sqrt{x^2-1}}{2}=2$$
$$\sqrt{x^2-1}=4$$
$$x^2-1=16$$
$$x^2=17$$
$$x=\sqrt{17}\ \ or\ \ -\sqrt{17}$$

Both are valid solutions of the original equation.

Matters are pedagogically tricky here as we first have students solve algebraic equations with actions that do not change solution sets, and so this point is moot. Forcing young students to "check their solutions" as a matter of habit at this level is pedantic and irritating. But matters are different in upper school work, and a discussion of the subtleties outlined here is absolutely necessary. (And now students must "check their solutions" as a matter of important habit and course!)

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