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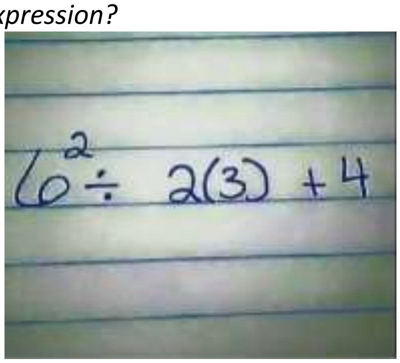
TANTON'S TAKE ON …

PEMDAS ON THE INTERNET

SEPTEMBER 2015

Every now and then I am asked to chime in on a mathematical debate that is making the internet rounds. Here's the latest one that came my way (though I feel I had seen it years before too).

What is the value of the following expression?



Is the answer 10 *or is the answer* 58*?*

(I can't track down the author of this. It came to me through FaceBook via this page:

I am impressed if the 95 WILL ROCK Morning Show really does discuss mathematics with its audience!)

Here are the two ways folk interpret the expression:

$$6^2 \div 2(3) + 4 = 36 \div 6 + 4 = 6 + 4 \quad = 10$$

$$\begin{aligned} 6^2 \div 2(3) + 4 &= 6^2 \div 2 \times 3 + 4 \\ &= 36 \div 2 \times 3 + 4 \\ &= 54 + 4 \qquad = 58 \end{aligned}$$

It seems from the comments posted that many people first have trouble seeing 58 as a possible answer. But the strict adherers of PEMDAS (parentheses, exponents, multiplication, division, addition, and subtraction) point out that in the order of operations rules, division and multiplication have "equal weight" and are to be calculated left to right when strung together. Since $2(3)$ means 2×3, we have in the expression the term

$$36 \div 2 \times 3$$

(the "E" makes sure $6^2 = 36$ is calculated first), which is to be calculated as 54.

MY REACTION

When I looked at the expression I calculated it as 10. But I knew there were troubles. (Well, I was told there were!) I had an emotional reaction too, one of deep hesitation, as I knew I was about to be drawn into a "Is a parallelogram a trapezoid?" type debate. Any discussion like this, including evaluating the expression, is not one about mathematics, but one of interpreting vague, varied, and inconsistent wordings of social conventions. It is impossible to argue with precision when the premise if fraught with imprecision.

> **Curious Example:** What does PEMDAS have to say about evaluating the following?
>
> $$\frac{10}{2+3}$$
>
> Division before addition?

More on this curious example in a moment.

THE TRUE RESPONSE TO $6^2 \div 2(3) + 4$:

Any discussion on evaluating this expression really should be moot. It is an example of a bad piece of writing and its author is at fault for writing a statement that can cause confusion and debate (even if the author thinks the reasons for confusion are unjustified). Absolute clarity in mathematics is the responsibility of a writer. (I know I fail regularly on this!) The editor of this mathematics piece should insist that the author rewrite the expression.

> **A Common Example of Bad Writing:**
> I have seen in algebra textbooks expressions of the form $a/2b$. As I'll explain in a moment, I personally assume the author means $\frac{a}{2b}$, with in-line version $a/(2b)$. But strict PEMDAS followers must interpret it as $\frac{a}{2}\times b$, that is, as $(a/2)b$.

But, of course, ambiguity and confusion was the intent of this author – especially with the placement of a space just after the *obelus* (the $\div$ sign). The author wanted our eyes to focus on the separate components 6^2 and $2(3)$ first.

The author also wanted to use confusing grade-school notation.

For me as a mathematician it is odd to see multiplication written as $2(3)$. I am aware this is common practice in the K-8 books, but parentheses mean something in mathematics, they mean grouping. We need them for $2(10-7)$, say (to avoid having folk think $2\times 10-7$), to tell readers that we intend to set 10 and -7 as a group together all to be pre-multiplied by 2.

In writing $2(3)$, however, there is nothing to be grouped and this comes across as just odd.

Comment: I suspect elementary school curriculums introduce parentheses as notation for multiplication to set the scene for later work in pre-algebra. There they will want students to write arithmetic details out in full:

$$2(10-7)=2(3)=6.$$

Quantities such as $2(3)$ then appear.

But trouble comes even later on in high-school when students are introduced to function notation. After years of all this training of course students are right to think that $f(x)$ means $f\times x$!

Secondly, the obelus, $\div$, is rarely used as a symbol of division in upper mathematics. It is now linked with another symbol, the horizontal bar, called the *vinculum*, which is an old symbol used for grouping. (More on this in a moment too. That now makes for three things to come back to!) So when I, as a mathematician, see the division symbol, I see it as based on a grouping symbol, with one group placed on top and one group based on bottom. (The dots in $\div$ represent this.) For example, $\frac{6^2}{2(3)}$ is the piece $6^2=36$ divided by the piece $2(3)=6$.

It is writing $6^2\div 2(3)$ that is confusing.

(Notice that the author of the expression deliberately avoided writing $36\div 2\times 3$.)

Strict PEMDAS followers will say that $6^2\div 2(3)+4$ is 58, and will have the technical law on their side. It is us followers of the vinculum that struggle. (Again, what is the value of $\frac{10}{2+3}$ in PEMDAS world?)

THE VINCULUM

Throughout the 1400s up to the 1700s before mechanical type machines were invented, it was the convention to use a horizontal bar, a *vinculum*, to denote grouping. Thus $2(10-7)$ was written $2\times\overline{10-7}$. (Those who know me know I adore the vinculum and am seriously pushing for its return to the curriculum!) An expression such as

$$3+\overline{220\div\overline{\overline{4\times 5}+2}}$$

is easy to evaluate (the answer is 13) and there is no ambiguity as to how one should evaluate it: always start with the inner-most vinculum first and work out from there.

Whatever the version of "PEMDAS" was 500 years ago, it began with a "V" – work out the vinculums first. No ambiguity occurs if vinculums are drawn over just two terms at a time and so in this context all one needs is "V."

Some Vinculum Simplifications:

No grouping is needed in strings of just addition: no matter how you place vinculums over the expression $2+3+4+5$, for example, the answer is 14 each and every time:

$$\overline{\overline{2+3}+4}+5=14$$
$$2+\overline{\overline{3+4}+5}=14$$

Additions can be added in any order one pleases and no vinculums are needed just for strings of additions. Writing $2\times\overline{3+6+2+1}$, for instance, is clear (it is 24).

The same is true for strings of multiplications. For instance, $\overline{2\times 3\times 5\times\overline{1+2+3}\times 10}+1+2$ is, without ambiguity, 1803. (Compute the product $2\times 3\times 5\times 6\times 10$ in any order you like.)

Writing polynomials is a little awkward with vinculums, even with these stated simplifications. Consider, for instance:

$$\overline{x\times x\times x}+\overline{3\times x\times x}+\overline{2\times x}+1.$$

With exponential notation for repeated multiplication, we can make this a little easier to read.

$$\overline{x^3}+\overline{3x^2}+\overline{2x}+1.$$

The vinculums here still look clunky, so let's make the convention to not bother drawing vinculums over multiplications. Thus we have the convention:

All multiplications come equipped with invisible vinculums. (Feel free to draw them back in if you like.)

With this convention, our polynomial now reads x^3+3x^2+2x+1.

Strings of subtractions require no vinculums either if no grouping is involved. For instance, $2-3-5-7$ is the string of additions $2+-3+-5+-7$ and so can be computed in any order. (However, $2-3-\overline{5-7}$ means something different.)

Divisions are best handled by avoiding the isolated vinculum-related symbol $\div$. Just use the actual vinculum itself!

For example, in the expression $\frac{10x+4}{3+7}$, I see a vinculum. This means the group—the entire group!—$10x+4$ is to be divided by the entire group $3+7=10$, so

$$\frac{10x+4}{3+7}=\frac{10x+4}{10}=x+\frac{4}{10}.$$

Pedagogical Aside: Let's bring back the vinculum! Understanding of the vinculum completely obviates this student confusion.

Common error:

$$\frac{\cancel{10}x+4}{\cancel{10}}=x+4$$

Response: Don't violate my vinculum!

Curious Example Answer: Consider again:

$$\frac{10}{2+3}$$

With understanding of the vinculum, this is clearly $10/5=2$. PEMDAS followers must argue that there are an invisible set of parentheses around the denominator. Might this, to say the least, be a tad confusing for our students? (Especially since this is never articulated and no-one ever writes those parentheses. No wonder the rigid "rules" in mathematics are mysterious and unclear.)

Comment: GEMA is now becoming popular: Evaluate all grouping symbols, then all exponents, then all multiplications (and divisions), and then all additions (and subtractions).

A Common Example of Bad Writing: In the expression $a/2b$ I see the symbol "/" which I interpret as an attempt to write a vinculum in line. So, with use of this symbol I must assume grouping is involved. The only grouping that can occur is with the 2 and the b. So, logically, I assume then $a/2b$ is meant to read as $\frac{a}{2b}$. (But PEMDAS follows must assume it is $(a\div 2)\times b$.)

The Vinculum Rules:
Here are the social rules of vinculum use we've come to:

1. *There is no need to write lots of vinculums over strings of additions and subtractions. They can be computed in any order one likes.* (PEMDAS folk insist on left to right.) *Just one big vinculum will do.*

2. *There is no need to write vinculums over strings of products. They can be computed in any order one likes. Just one big vinculum will do.*

3. *Actually… there is no need to draw the vinculum over a string of products/exponents. It will be understood that they come with invisible vinculums.*

4. *Avoid the* $\div$ *symbol. Just use the actual vinculum!*

5. *Keeping invisible vinculums in mind (feel free to draw them in if you like), compute by starting with inner-most vinculums first and working outwards. Vinculums that are equally nested can be worked out in any order.*

For example,

$$\frac{\dfrac{\overline{2+3}+\overline{1+9}}{2^2+1}-3+2\times 5}{1-\dfrac{\overline{2-\dfrac{5}{10+2}\times\overline{2\times 4+4}+3}}{17+\dfrac{1}{1+2}}}$$

is 10. (The terms $\overline{2+3}$ and $\overline{1+9}$ are equally nested.)

THE VINCULUM TODAY

Even though the vinculum is no longer discussed today, it does still exist as notation for grouping in some modern contexts. Fraction operations, as we have seen, use the vinculum:

$$\frac{2a^2+ab+3}{a}=2a+b+\frac{3}{a}.$$

The *radix* symbol √ for square roots is now usually accompanied with a vinculum. For example, writing √9+16 is unclear. But matters are clear if you include the vinculum so show the intended grouping:

$$\sqrt{9}+16=19$$
$$\sqrt{9+16}=5$$

Comment: Most people today don't realise that $\sqrt{}$ is a combination of two symbols.

In writing repeating decimals, we use a vinculum to indicate which group is repeated. For example, $0.5\overline{18}$ means $0.51818181818\ldots$.

Even in geometry the notation $\overline{AB}$ means the points A and B are "grouped together" via a line segment. (This is not purely coincidental notation.)

FINISHING UP
As an unrelated, but important, aside, let's answer a list of annoying questions once and for all.

Is a parallelogram a trapezoid?
Is a square a rectangle?
Is an equilateral triangle isosceles?
Is 1 prime or is it not?

The reason these questions are annoying and keep coming up and drive many an educator batty is that there is no universal definition for many of these things: their definitions vary from textbook author to textbook author. Yet standardized tests assume all is set and agreed upon. That's a worry!

Here are two definitions I have seen in textbooks for trapezoid.

Definition 1: A *trapezoid* is a quadrilateral with a pair of parallel sides.
Definition 2: A *trapezoid* is a quadrilateral with exactly one pair of parallel sides.

The first definition is a tad vague: What if the quadrilateral, a parallelogram, has two pairs of parallel sides?

Well, reading definition 1 like a lawyer, a parallelogram certainly has "a pair of parallel sides" and so fits the bill. According to definition 1, parallelograms are trapezoids.

Definition 2 is clear in its statement, and excludes parallelograms from the class of trapezoids.

Which is the correct definition? No idea.

Is it a big deal?

To a mathematician, no. If a mathematician ever needs to talk about trapezoids in a paper she is writing, being a good writer, she will make it clear which definition she is using in her work.

Is it a big deal to a student taking a national test? Yes! One needs to look up the national test authors' definition.

Definition: A *rectangle* is a quadrilateral with four right angles. ("Four *recht* angles")

This seems to be standard definition, in which case squares fit the bill of being rectangles too.

Definition: A triangle is called *isosceles* if it has two sides equal in length.

Reading the definition like a lawyer, equilateral triangles do indeed have "two sides equal in length" and so qualify as being isosceles. (It would be better to write "at least two sides" in the given definition.)

Mathematicians have a reason for wanting to exclude 1 from the list of prime numbers: to stop prime factorizations, that is, factor trees, from going on forever.

$$\begin{aligned} 20 &= 4\times 5 \\ &= 2\times 2\times 5 \\ &= 1\times 2\times 2\times 5 = 1\times 1\times 2\times 2\times 5 = \cdots \end{aligned}$$

To this end, mathematicians use the following definition of a prime number:

Definition 1: A number is *prime* if it has exactly two factors.

This, however, is not the definition presented in most school books.

Definition 2: A number is *prime* if its only factors are 1 and itself.

According to the author of definition two, 1 is a prime number and so must be considered prime throughout that author's textbook.

But some authors might qualify definition 2 by adding to the definition that the number must be larger than 1. (And so, to them, 1 is not prime.)

Comment: Teaching students to read definitions like a lawyer is a valuable exercise in and of itself. Why not look at and analyse a series of definitions from different textbooks and sources with your students? How enlightening for all that would be!

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New Twitter Hashtag:

#bringbackthevinculum