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TANTON'S TAKE ON …

THE WORD "OF"

OCTOBER 2014

The word "of" is a very confusing word. We often use it in everyday language when speaking of quantities or relations between quantities and, as such, it often comes up in the mathematics classroom.

Consider these six statements. Is the meaning of "of" the same in each?

> *The third* ***of*** *five children.*
> *A third* ***of*** *twelve.*
> *Six groups* ***of*** *three.*
> *Nine out* ***of*** *ten dentists.*
> *I'll take two* ***of*** *the three.*
> *85%* ***of*** *Australian men.*

We often tell students that "of" translates to the action of multiplication. Clearly that aphorism does not apply in all contexts!

Comment: One really should try to avoid blanket "if you see this … do this …" statements in teaching. Education is about helping students examine the contexts of claims and letting them decide for themselves the appropriate actions to take.

Consider, for example, the following four questions:

a) What is 20 take away 10 ?
b) My history teacher wants us to read pages 10 through 20 for homework tonight. How many pages of reading is this?
c) I am tenth in line. My friend is twentieth in line. How many people are between us?

d) Ulrike was born in year 2010. Today is October 1st, 2020. If asked "How old are you?" might Ulrike legitimately answer "Nine years old"? Might she legitimately answer "Ten years old"? How about "Eleven years old"?

There is no general blanket procedural statement here. One can only pause and mull over each statement individually to then decide how best to proceed in each case. (Was that, in and of itself, a blanket statement?)

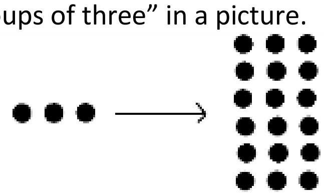
It is true that multiplication is often the appropriate arithmetic translation of "of" in statements.

For example, at the very elementary level, the multiplication of whole counting numbers is defined to be repeated addition:

$$\begin{aligned} 6\times 3 &= \text{six groups of three} \\ &= 3+3+3+3+3+3 \ . \end{aligned}$$

The word "of," in this context, is inherently linked with multiplication.

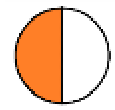
Question: Geometrically, repeated addition corresponds to repeating a picture representing a quantity. Here's "six groups of three" in a picture.

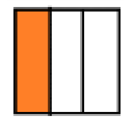


Suppose we focus instead on the unit of measurement used in the picture and repeat that unit six times.

Can we say that geometric scaling is, in some sense, repeated addition too?

Another use of "of" in early mathematics sits with "parts of a whole." We talk of "half of a pie" or "a third of the people in the room" and link "of" with a concept of division.

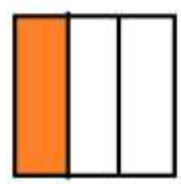
Half of a round pie =  .

A third of a square =  .

But in these discussions one always needs to be on the lookout for what is being divided – what is the "whole" implied? Statements that involve compound division actions are subtle in this regard. Consider, for example, the notion:

Half of a third of a square pie.

The whole associated with the word "half" is one third of the pie, with the whole for that one-third being the full square pie. There are two different "wholes" in this one statement.



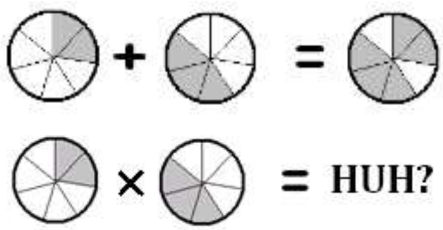
one-third of the whole square

one-half of the one-third of the whole square

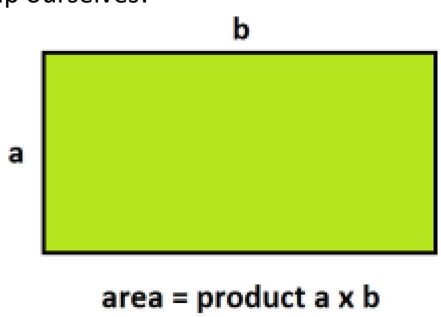
We then ask students to notice that this matches one-sixth of the full square. (Using the first of the previous two "wholes.")

"OF" AND FRACTIONS

As discussed in the March 2014 Curriculum Essay "Fractions are Hard!" (see www.jamestanton.com/?p=1072), fractions are hard! When viewed as portions of pie (as they are often presented to students), it makes perfectly good sense to add fractions. But it makes no sense whatsoever to multiply fractions!



Yet pictures of shaded portions of squares, as shown on the previous page, compel us to think "area." And when we think area, we naturally think multiplication. We can't help ourselves!



The lure of this is strong. We abandon the pie model for fractions and feel compelled to believe that we can multiplying fractions after all.

Let's look at a specific example, in laborious detail, keeping laborious track of the notion "whole" as we go along. We'll see how confusing it all really is.

EXAMPLE: Interpret and compute: *Two-fifths of three-sevenths.*

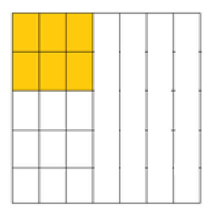
OVERLY DETAILED ANALYSIS: We have two-fifths of something, of some whole. Whatever that whole is we need to divide it into five equal parts and select two of them. That will be our two-fifths.

But what is the whole for that two-fifths? It's an abstract three-sevenths. This should be three-sevenths of something else, but that something else is not mentioned. What to do?

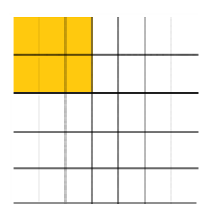
People usually use squares to represent unmentioned wholes. Let's do so here too.

The square is the whole for the three-sevenths. We divide the square into seven equal parts, and select three of them. Those three parts are the "whole" for the two-fifths.

So divide this three-sevenths into five equal parts and select two of them. This gives us the two-fifths of three-sevenths (of our unspecified whole).



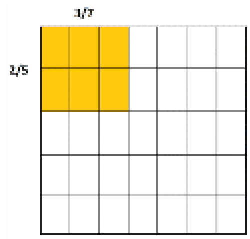
We now see the square divided into thirty-five equal parts, with six of those parts highlighted. This is, in and of itself, a picture of $\frac{6}{35}$ (back with the original square as the whole).



But we also see a shaded rectangle and feel compelled to think multiplication. We are drawn to the call of "length times width."

To follow this call, we need to change our context of "whole" yet again.

We see that the left edge of the square has been divided into five equal parts (so the "whole" here is the left edge) with two parts highlighted. The top edge (yet another "whole") of the square has been divided into seven equal parts with three highlighted. We notice $\frac{2}{5}$ and $\frac{3}{7}$.



And because of our predilection for area, it seems natural to interpret this as a picture of the product $\frac{2}{5}\times\frac{3}{7}$, the area of a $2/5$ -by-$3/7$ rectangle. And since the picture represents the fraction $\frac{6}{35}$, we feel compelled to <u>declare</u> the product of the two fractions to be:

$$\frac{2}{5}\times\frac{3}{7}=\frac{6}{35}\,!$$

SUMMARY OF WHAT HAS JUST HAPPENED:
1. *The multiplication of fractions has no meaning.* (You can't multiply pieces of pie or proportions in and of themselves!)

2. *We feel that "two-fifths of three-sevenths" has meaning.*

3. *To interpret "two-fifths of three-sevenths" we need to be flexible of what we mean by the "whole" and keep changing contexts for it.*

(In the example the word "whole" had four different meanings: it was the full unit square (to get to three-sevenths), it was the three-sevenths itself (to get to two-fifths of three-sevenths), it was the left edge of the square (to see $\frac{2}{5}$ explicitly in the diagram), and it was the top edge of the square (to see $\frac{3}{7}$ in the diagram). Whoa!)

4. *The area model suggests that a consistent way to define the product of two fractions is to interpret the product as a fraction of a fraction of some unit whole. The area model shows how to actually carry out and interpret the computation.*

The point is that we use the area model to <u>define</u> the product of fractions. The product is interpreted both as a "fraction of a fraction" and as a portion of area. As such, in this setting the word "of" is outright declared to be linked with the action of multiplication!

(This is all abstract and philosophically very hard and confusing. Did I mention that fractions are hard?)

We never actually say in the curriculum what a fraction truly is. The interpretation of "whole" constantly changes and we keep showing students very different models to explain different mechanics of fractions, different features of fraction-ness we like to believe are true. With contradictory messages like:

> *Fractions are a portion of a whole.*
> *Here's how you multiply fractions.*

or

> *Fractions are numbers on the number line.*
> *Here's how you multiply fractions: interpret the final picture as a portion of the unit square.*

students' (and our own) true understanding of fractions can only be vague and hazy at best. It is easy to teach the mechanics of fractions, but it is hard to really internalize what a fraction truly is. (Again, see the March 2014 Curriculum Essay.)

Nonetheless, by declaration of the area model, the word "of" in a statement of the form *"a fraction of a fraction"* does translate to the mathematical action of multiplication.

> **Question:** In the statement
>
> *A third of twelve*
>
> is there are declared whole of "twelve"? (A third of twelve is four.) Is the area model at play here?

WRITERS' AND TEACHERS' RESPONSIBILITY

In the context of elementary arithmetic and work with fractions, the word "of" is defined to be linked with the action of multiplication. (Three of the six opening examples presented can be thought of this way.) But not all uses of the word "of" operate in this manner.

We do have a societal convention to interpret "of" as a call for multiplication in statements that mention fractions. (For example, "half of six" is $\frac{1}{2}\times 6=3$.) But without mention of a fraction, societal conventions are unclear. (How should we interpret "two of six"? Mathematical uniformity would suggest this be $2\times 6=12$.)

Because there is no general procedure for interpreting the word "of" in sentences, it is the responsibility of the author of any "of" statement - be that author a textbook writer, a teacher writing a problem, or a student writing an answer - to clarify intent and obviate ambiguity if it could exist.

And teachers must always encourage students to pause over statements, to reflect on possible interpretations, and to then select appropriate courses of mathematical action.

Such work is subtle and difficult to conduct, but it is of(!) mighty good importance. Discussing and being direct about sources of confusion helps obviate the confusion.

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