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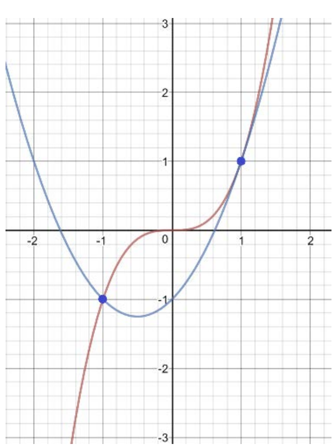
TANTON'S TAKE ON …

TANGENT LINES AND TANGENT CURVES

MAY 2018

Here's a challenge:

Find the equation of a parabola that intersects the cubic $y = x^3$ *at* $(-1,-1)$ *and is tangent to the curve at* $(1,1)$.



Here's another one.

A line through $(5,125)$ *is also tangent to the curve* $y = x^3$ *at some point. Which point?*

And another!

What is the equation of the tangent line to the curve $y = x^5$ *at* $x = k$ *?*

By the way, answer these questions using only the tools of Algebra II!

A colleague Alma Teao Wilson recently pointed out to me a simple idea that is extraordinarily powerful.

Consider two functions f and g related this way, for example:

$$g(x) = f(x) + (x-2)^2 (x-3)(x-4)^8.$$

Then we have that $g(2) = f(2)$, $g(3) = f(3)$, and $g(4) = f(4)$, and so the graphs of the two functions intersect at $x = 2$, $x = 3$, and $x = 4$. Moreover, for values x close to 2 we have that

- $(x-2)^2$ is positive
- $(x-3)(x-4)^8$ has value close to $(2-3)(2-4)^8 = -256$

and so

- $g(x) = f(x) + (negative\ value)$

for values just either side of $x = 2$. That is, the graph of $y = g(x)$ is below the graph of $y = f(x)$ near $x = 2$, with $g(x) = f(x)$ right at $x = 2$. That is, the two graphs are tangent at $x = 2$.

In the same way, the two graphs are tangent at $x = 4$, but the two graphs cross at $x = 3$. (We see $g(x) - f(x)$ changes sign at $x = 3$.)

In general: If

$$g(x) = f(x) + (x-k)^n \times (something)$$

with the value of "something" in the parentheses non-zero at $x = k$, then

- $g(k) = f(k)$
- The sign of $g(x) - f(x)$ does not change at $x = k$ if n is even (and does if n is odd).

Thus the graphs of f and g intersect at $x = k$ and are tangent there if n is even.

The opening question asks to find the equation of a parabola tangent to $y = x^3$ at $x = 1$ and to intersect the graph at $x = -1$.

For any value a we have that

$$x^3 + a(x-1)^2 (x+1)$$

is a curve intersecting $y = x^3$ these two ways. Now choose a value for a so that, upon expanding, the x^3 terms cancel to leave a degree-two equation. We see that $a = -1$ does the trick, giving the quadratic

$$\begin{aligned} y &= x^3 - (x-1)^2 (x+1) \\ &= x^2 + x - 1 \end{aligned}$$

A line though $(5,125)$ also tangent to $y = x^3$ at some point $x = b$ has the form $x^3 - (x-5)(x-b)^2$, that is, the form

$$(5+2b)x^2 - b(10-b)x + 5b^2$$

and so must have $b = -2\frac{1}{2}$ to be a line!

Lovely!

Exercise: *Find the equation of the tangent line to the curve $y = x^4 - 2x + 1$ at $x = 3$.*

Start to the Answer: Consider

$$y = x^4 - 2x + 1 + (x-3)^2 (ax^2 + bx + c).$$

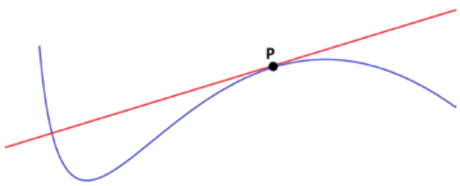
The graph of this function is tangent to the given curve at $x = 3$. Can one now choose values for a, b, and c which leaves a degree-one equation upon expansion?

This approach shows that we can compute equations of tangent lines to (polynomial) curves without calculus!

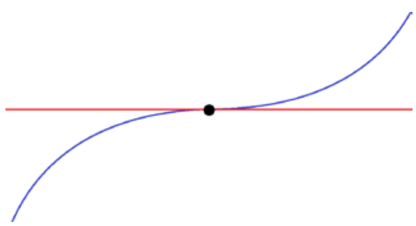
Caveat: We have to be careful here. It seems we are operating with the following definition of a "tangent line."

A line is tangent to a curve at a point P if the curve and the line meet at P, and all points on the curve just near P lie on just one side of the line.

(Tangent lines may cross through curves away from the tangency points.)



With this definition there is no tangent line to the curve $y = x^3$ at the origin. Yet calculus says to regard the horizontal as tangent to the curve at $x = 0$.



(So what is the calculus definition of a tangent line?)

Exercise: Prove that, indeed, there is no line $y = mx$ through the origin with the property that all points on the curve $y = x^3$ near the origin lie on one side of the line.

Despite the caveat, one can use this algebra to motivate calculus thinking and work. We can define the slope of a curve at a point to be the slope of the tangent line to the curve at that point. And to find the equation of the tangent line (and hence its slope) simply work it out!

Example: *a) Show that the equation of the tangent line to the curve $y = x^5$ at $x = k$ is $y = 5k^4x - 4k^5$.*

b) In general, show that the slope of the tangent line to $y = x^n$ at $x = k$ is nk^{n-1}.

Answer: a) We can work with

$$y = x^5 + (x-k)^2\left(ax^3 + bx^2 + cx + d\right)$$

and choose appropriate values for a, b, c, and d so that this equation reduces to a linear equation.

Comment: Students can do this for $y = x^2$, $y = x^3$, $y = x^4$, and so on, and identify structure to the algebraic work.

b) Let me be slick and a bit clever here. With the binomial expansion we have

$$\begin{aligned} y = x^n &= (x-k+k)^n \\ &= (x-k)^n + nk(x-k)^{n-1} + \cdots \\ &\qquad + \binom{n}{2}k^{n-2}(x-k)^2 + nk^{n-1}(x-k) + k^n \\ &= (x-k)^2\left((x-k)^{n-2} + nk(x-k)^{n-1} + \cdots + \binom{n}{2}k^{n-2}\right) \\ &\qquad + nk^{n-1}(x-k) + k^n \\ &= (x-k)^2(\mathit{something}) + nk^{n-1}(x-k) + k^n \end{aligned}$$

with the "something" non-zero at $x = k$.

This shows me that

$$x^n - (x-k)^2(\mathit{something})$$

is a curve tangent to $y = x^n$ at $x = k$ and that actually this curve is a line: it is the line

$$y = nk^{n-1}(x-k) + k^n$$

of slope nk^{n-1}.

Exercise: *a) If f and g are polynomials, prove that the slope of $y = f(x) + g(x)$ at $x = k$ matches the sum of the slopes of each of $y = f(x)$ and $y = g(x)$ at $x = k$.*

b) What other standard differentiation rules can be proved purely algebraically for polynomials?

Of course, a number of questions arise.

1. Is there a way to handle the situation when the "something" is zero at $x = k$? For example, does our algebraic work determine the slope of the tangent line to the curve $y = x^n$ at $x = 0$ at least for n even?

2. Can we generalize our geometric definition of a tangent line to have context and meaning for all polynomial curves at all points (such as, at the origin for $y = x^3$)?

3. Can we compute tangent lines to non-polynomial curves at points, such as, for $y = \sqrt{x}$ or $y = \frac{1}{x}$ or $y = \sin(x)$?

Question: What do you think of

$$\sqrt{x} + \frac{1}{2\sqrt{b}}\left(\sqrt{x} - \sqrt{b}\right)^2$$

for $b > 0$?

4. What happens to this algebraic method if we push it too far? For example, can we find a parabola that is tangent to the $y = x^3$ at two distinct locations? Can we find equations of tangent lines to $y = x^3$ that happen to pass though the origin?

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