CURRICULUM INSPIRATIONS: www.maa.org/ci

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TANTON'S TAKE ON …

IRRATIONAL NUMBERS

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Everyone just "knows" that π is irrational, as are most square roots: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, ….. (Why did I skip over $\sqrt{4}$ and $\sqrt{9}$ in this partial list?)

After all, many (and I mean many) textbooks and curricula have assessment questions of the type:

Question 1: Which of the following numbers are irrational?

a) $\sqrt{64}$ b) $\sqrt{11}$ c) $\frac{73}{37}$ d) $\frac{\pi}{2}$

And everyone "knows" that irrational numbers have infinitely long decimal expansions that never fall into a repeating pattern. I've seen many assessment questions of this type too:

Question 2: Which of the following numbers are irrational?

a) 0.132
b) $0.\overline{81}$
c) $6.55702702702702...$
d) $0.79162...$

And when I see questions such as these, I cry – at least internally.

A rational number is (or at least should be) defined as any number equivalent to a number of the form $\frac{a}{b}$, with a and b integers, b nonzero. Thus $5\frac{1}{2}$, $\frac{3\frac{5}{7}}{7\frac{9}{19}}$,

$\frac{0.02}{1.872}$, $\sqrt{\frac{0.01}{441}}$, and $\frac{\sqrt{2\pi}}{\sqrt{8\pi}}$, for example, are definitely rational numbers. (Another word for a rational number is *fraction*.)

And one can prove from the process of long-division that computing the decimal expansion of a rational number $\frac{a}{b}$, thought of as $a \div b$, has you fall into a cycle of repeating remainders. (If you fall into repeating zeros, then we usually say that the decimal expansion of $\frac{a}{b}$ is finite and do not bother to write the trail of infinitely many zeros.) Lessons 2.4 and 2.2 of www.gdaymath.com/courses/exploding-dots/ for example, explain the details.

From this we conclude:

Any real number that has an infinitely long decimal expansion that does not fall into a repeating pattern cannot be rational.

For example, consider the real number

$$x = 0.1010010001000010000001....$$

with the implied pattern continuing. Although the decimal expansion for x has a pattern to it, it is not a repeating pattern. This means that x cannot be a fraction. That is, x is an irrational number.

This establishes that irrational numbers actually exist. (And now it is fun for students to invent other examples of irrational numbers for themselves: $0.1020304050607080901001101201 30...$ for example.)

Conversely, one can prove that any number with a repeating decimal expansion is actually rational. This is usually established as follows:

Example: Show that $0.8\overline{13}$ is rational.

Answer: Let $0.8131313.... = w$.
(I chose the letter w for "I don't know <u>w</u>hat it is.")

Then we have:

$$0.81313131.... = w$$
$$8.13131313.... = 10w$$
$$81.3131313... = 100w$$
$$813.131313... = 1000w$$

We see that

$$10w = 8 + 0.131313...$$
$$1000w = 813 + 0.131313....$$

So $990w = 805$ and

$$w = \frac{805}{990}.$$

It is rational.

Example: Show that $0.57 = 0.57\overline{0}$ is rational.

Answer: Easy.

$$0.57 = \frac{57}{100}.$$

These two examples generalize to establish that all decimals with a repeating pattern (which includes the finite decimals) do correspond to rational numbers. The best curriculums I have seen do have students work through these ideas too. But many don't, assuming students just "know" these facts about patterns and non-patterns in decimal expansions.

But even many of the best curriculums miss the opportunity to point out, with this understanding in hand, that we can now see that irrational numbers exist: the number $x = 0.1010010001...$ described earlier, for example, must be irrational.

SO, WHERE DO WE CURRENTLY STAND?
Right now we have the means to answer assessment questions 1a), 1c), 2a), 2b), and maybe 2c) and 2d).

The answers so far:

1a) $\sqrt{64} = 8 = \frac{8}{1}$ is rational.

1c) $\frac{73}{37}$ fits the definition of being rational.

2a) and b) are each repeating decimals and so are each rational.

2c) **IF** we can assume the pattern of repeating 702 s continues, then we have a rational number. If we can't assume this, then… who knows? Maybe it continues to repeat or maybe it gets chaotic later on?

2d) Who knows? Maybe the number is $0.7916\overline{25507}$, in which case it would be rational. Or maybe the decimal continues as $0.79162101001000100001000001...$ (with the pattern implied continuing), in which case the number is irrational. There is not enough information to answer this question.

As for questions 1b) and d), we have absolutely no means to answer them!!

A FAULTY CURRICULUM ATTEMPT
Some curriculums do attempt to lead students to the conclusion that $\sqrt{2}$ and π , for instance, are indeed irrational. They might show, for example, the first one-hundred digits of π :

3.1415926535897932384626433832795028
84197169399375105820974944592307816
4062862089986280348253421170679…

There are no repeating patterns so π must be irrational? Hmm.

What about the first one-hundred digits of $\frac{352}{541}$ then?

0.6506469500924214417744916820702402
95748613678373382624768946395563770
7948243992606284658040665434381…

Again I see no repeating pattern. So this means $\frac{352}{541}$ is irrational too?

As our answer to assessment question 2d) points out, just looking at the first few decimals in the decimal expansion of a number (be that the first five digits, the first one-hundred digits, or the first quadrillion-and-two digits) gives you no clue as to whether or not a number is rational or irrational. We cannot ascertain the status of π from its first one-hundred decimal digits. (By the way, the decimal expansion of $\frac{352}{541}$ does, of course, eventually repeat. You need to head out beyond the 540 th digit to see the repeat.)

BIG WARNING

The remainder of this essay is devoted to establishing that assessment questions 1b) and 1d) are completely and utterly wrong to ask of students. We will prove now that $\sqrt{11}$ and π , and hence $\frac{\pi}{2}$, are indeed irrational. I will offer here the simplest proofs I know of these facts and you will see how significantly far beyond the tools of the standard curriculum these proofs lie.

My point is: **Asking students about the irrationality of π and the roots of non-perfect powers for assessment purposes is pedagogically wrong.**

Just to be clear: I have no trouble with taking time to discuss with students the story of man's struggle to prove the irrationality of roots. There are, for example, lovely accessible visual proofs of the irrationality of $\sqrt{2}$, and juicy tales of Hippasus losing his life for the discovery of its irrationality. (The internet offers all details.)

And also I have no trouble telling the tale of mankind's struggle to prove that π is irrational. It took mathematicians literally millennia to establish this fact after many false starts. (The first proof appeared in the mid-1700s and was offered by Swiss mathematician Johann Lambert. The argument I present below is essentially the approach he took.)

But assessing students on these irrationality claims boils down to assessing: **Have you memorized some facts I just told you?** We can, and must, do better than that in an enlightened curriculum.

Okay … to the proofs:

Example: Establish that $\sqrt{11}$ is irrational.

Answer: Let's be contrary. Let's assume that $\sqrt{11}$ is rational and see what goes wrong with the mathematics under this assumption.

Suppose $\sqrt{11} = \frac{a}{b}$ for some integers a and b. Then we must have $11 = \frac{a^2}{b^2}$, and so $a^2 = 11b^2$.

The Fundamental Theorem of Arithmetic, thanks to Euclid (ca. 300 BCE), states that every integer can be written as a product of primes in just one way – if we order the primes in the product from smallest to largest.

(See www.jamestanton.com/?p=680 and www.jamestanton.com/?p=695 for details.)

So this means the number a is a product of primes, $a = 2^{a_2}3^{a_3}5^{a_5}7^{a_7}11^{a_{11}}13^{a_{13}}\cdots$, as is b, $b = 2^{b_2}3^{b_3}5^{b_5}7^{b_7}11^{b_{11}}13^{b_{13}}\cdots$. (The majority of the exponents in these expressions are zero.) Thus the equation $a^2 = 11b^2$ reads:

$$2^{2a_2}3^{2a_3}5^{2a_5}7^{2a_7}11^{2a_{11}}13^{2a_{13}} = 2^{2b_2}3^{2b_3}5^{2b_5}7^{2b_7}11^{2b_{11}+1}13^{2b_{13}}\cdots$$

So we now have a number, a^2, which is the same as $11b^2$, with two different-looking prime factorizations. As a number can only have one prime factorization it must be the case that the exponents match:

$$2a_2 = 2b_2$$
$$2a_3 = 2b_3$$
$$2a_5 = 2b_5$$
$$2a_7 = 2b_7$$
$$2a_{11} = 2b_{11} + 1$$
$$2a_{13} = 2b_{13}$$
$$\vdots$$

But look at the equation $2a_{11} = 2b_{11} + 1$. The left side is an even number. The right side is an odd number. These exponents simply cannot match!

Oops. The mathematics has broken down. It must be the case that our beginning assumption that $\sqrt{11}$ is rational is wrong. Therefore, $\sqrt{11}$ must be irrational.

Example: Establish that π is irrational.

Answer: Let's make full use of the tools of calculus, in particular, the tools of infinite Taylor series.

REMINDER: *What comes next really is the easiest proof I know!*

For each non-negative integer n let $P_n(x)$ be the following infinite series:

$$P_n(x) = 1 - \frac{1}{2n+1}\cdot\frac{1}{2}x^2 + \frac{1}{2n+1}\cdot\frac{1}{2}\cdot\frac{1}{2n+3}\cdot\frac{1}{4}x^4 - \frac{1}{2n+1}\cdot\frac{1}{2}\cdot\frac{1}{2n+3}\cdot\frac{1}{4}\cdot\frac{1}{2n+5}\cdot\frac{1}{6}x^6 + \cdots$$

For example,

$$P_0(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x,$$

$$P_1(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \frac{\sin x}{x}.$$

An exercise in algebra shows that:

$$P_n(x) = P_{n+1}(x) - \frac{1}{2n+1}\cdot\frac{1}{2n+3}\cdot x^2 \cdot P_{n+2}(x)$$

and so we have:

$$\frac{P_n}{P_{n+1}} = 1 - \frac{x^2}{(2n+1)(2n+3)\cdot\frac{P_{n+1}}{P_{n+2}}}.$$

(I hope you don't mind me dropping the (x) s for visual ease.)

This means that:

$$\frac{P_0}{P_1} = 1 - \frac{x^2}{1\cdot 3\cdot\frac{P_1}{P_2}}$$

$$= 1 - \frac{x^2}{3\left(1 - \frac{x^2}{3\cdot 5\cdot\frac{P_2}{P_3}}\right)}$$

$$= 1 - \frac{x^2}{3 - \frac{x^2}{5\cdot\frac{P_2}{P_3}}}$$

$$= 1 - \frac{x^2}{3 - \frac{x^2}{5\left(1 - \frac{x^2}{5\cdot 7\cdot\frac{P_3}{P_4}}\right)}}$$

$$= 1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7\cdot\frac{P_3}{P_4}}}}$$

$$= 1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \frac{x^2}{9\cdot\frac{P_4}{P_5}}}}}$$

and so on.

Now $\frac{P_0}{P_1} = \frac{\cos x}{\sin x / x} = x\cot x$ and Lambert managed to argue that the infinite tower of numbers we're heading to does indeed have a finite meaningful value. (That is, the sequence of values we're producing converges.) So we have established:

$$x\cot x = 1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \cfrac{x^2}{11 - \cfrac{x^2}{13 - \cdots}}}}}}$$

A word about towers of fractions:
Any finite tower of fractions unravels to a single rational number. For example:

$$1 - \cfrac{\frac{4}{9}}{3 - \cfrac{\frac{4}{9}}{5 - \frac{4}{9}}}$$

unravels to the fraction $\frac{75}{239}$.

It is also true that every rational number can be written as a finite tower of fractions, in essentially only one way. (This is not entirely trivial to establish, but it well known in the theory of "continued fractions.")

We thus conclude that any number that is represented as an infinite tower of fractions cannot be rational.

This means:

If x^2 is a rational number, then $x\tan x$ must be irrational.

All right. Now we're ready for the proof that π is irrational. Let's again be contrary and see what goes wrong with the mathematics if we assume that we can write $\pi = \frac{a}{b}$ for some integers a and b.

Let $x = \frac{\pi}{4}$. This will be rational too as it equals $\frac{a}{4b}$. And we also have that $x^2 = \frac{a^2}{16b^2}$ is rational. By the result we established from the tower of fractions, we must have that $x\tan x$ is irrational. But

$$x\cot x = \frac{\pi}{4}\cot\left(\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

So we have just established that $\frac{\pi}{4}$ is both rational and irrational! Oops!

The mathematics has broken down so it must be the case that our initial assumption that π is rational is wrong. We conclude that π is irrational after all.

Question: So where does this argument fit into the school curriculum? You see why I want to cry over those assessment questions.

tanton.math@gmail.com