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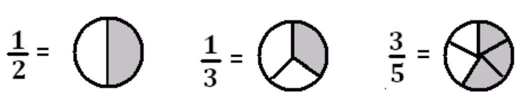
TANTON'S TAKE ON …

FRACTIONS ARE HARD!

MARCH 2014

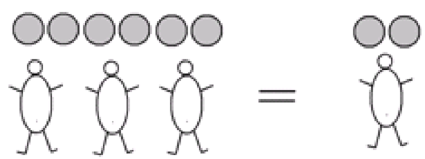
Let me just state at the outset that **fractions are hard!** Mankind has struggled with them for centuries - and rightly so - and we individually struggle with them for years, if not decades, and again, rightly so! It is completely unrealistic - unfair even - to expect students to be comfortable with fractions by the end of grade school, by the end of middle school, or even by the end of high school. If we think about how fractions are introduced and used throughout the standard curriculum, matters are fundamentally confusing and contradictory.

In the early grades, fractions are often introduced as pieces of pie, or parts of some other favorite whole.



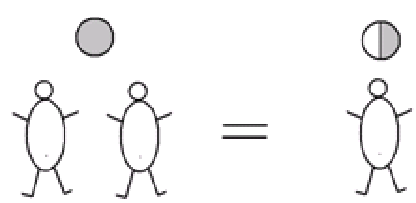
This is often motivated through sharing:

If 6 *pies are shared equally among* 3 *boys, how many pies does each individual boy receive?*



We write: $\frac{6}{3} = 2$ pies per boy.

If 1 *pie is shared equally between* 2 *boys, how much does each individual boy receive?*



We write: $\frac{1}{2} =$ half a pie per boy.

A Point of Confusion: So what is a fraction? Is it an amount of <u>pie</u> or an amount of <u>pie per boy</u>? In my science class we are very fussy about units. What are the units here?

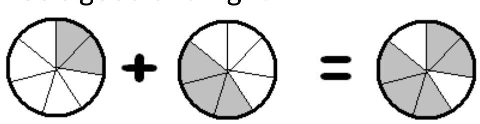
We have right off the bat two slightly different models for what a fraction is.

<u>Model 1</u>: *A fraction is an actual amount of pie one can physically handle.*
<u>Model 2</u>: *A fraction is a proportion of pie per boy.*

Often these models are presented as though they are interchangeable. But there is something unsettling, something hard to articulate, about doing this. Those who think like scientists feel particularly unsettled.

Nonetheless, each of these models is good at motivating certain features we feel ought to be true about fractions.

Model 1 is good for motivating the basic addition of fractions
If I am handling concrete pieces of pie, then drawing pictures of the following ilk feels good and right:



This motivates our belief about how we should add fractions: $\frac{2}{7}+\frac{3}{7}=\frac{5}{7}$. Here we are adding sevenths as though we are adding apples.

[Model 2, on the other hand, is very confusing here: What does $\frac{2}{7}+\frac{3}{7}$ mean? If 2 pies are being shared among 7 boys, and 3 pies are being shared among 7 seven boys, is that 5 five pies being shared among 14 boys? Or is it the same seven boys?]

Model 2 is good for motivating several fundamental fraction beliefs

In this model $\frac{a}{b}$ represents the amount of pie an individual boy receives when a pies are distributed among b boys.

Question : What's $\frac{10}{10}$? This is ten pies being shared equally among ten boys. That's one pie per boy. $\frac{10}{10}=1$.

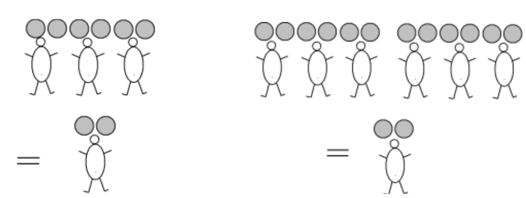
Question: What's $\frac{10}{1}$? That's ten pies being given to one boy. (Lucky boy!) That's ten pies per boy: $\frac{10}{1}=10$.

These suggest, in general, that $\frac{a}{a}=1$ and $\frac{a}{1}=a$.

Suppose we are sharing a pies among b boys. What happens if we double the number of pies and double the number of boys? Nothing! The amount of pie per boy is still the same:

$$\frac{2a}{2b}=\frac{a}{b}.$$

For example, as the picture shows, $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies for each boy.



Tripling the number of pies and tripling the number of boys does not change the final amount of pie per boy, nor does quadrupling each number, or one-trillion-billion-tupling the numbers!

$\frac{6}{3} = \frac{12}{6} = \frac{18}{9} = \cdots$ = two pies per boy

This leads to believe, in general, $\frac{xa}{xb} = \frac{a}{b}$ (at least for positive whole numbers a, b, and x).

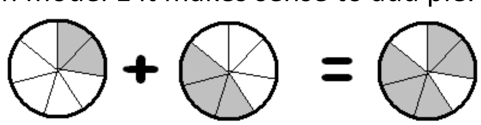
> **A Point of Confusion:** Fractions are particularly confusing to young students because they are the first type of number they encounter that are not represented in unique ways. For example, $\frac{15}{20}$ and $\frac{9}{12}$ are the same number even though completely different symbols are being used each time to represent it.

Comment: One can motivate the fraction belief $\frac{xa}{xb} = \frac{a}{b}$ using model 1.

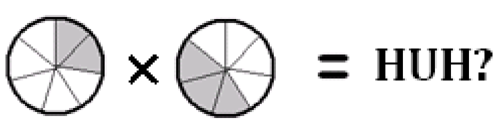
(Does that model also motivate $\frac{a}{1} = a$?)

WHERE BOTH MODLES FAIL

In model 1 it makes sense to add pie.



What does it mean to multiply pie?

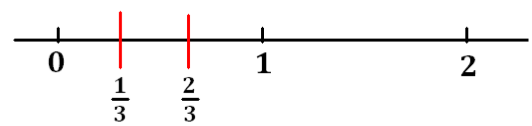


And I can't even begin to imagine what it means to multiply pies per boy!

After teaching students that fractions are related to pie, it is not uncommon for a curriculum to change track and introduce a third model for fractions, one that let's go of the pie model, but allows for the multiplication of fractions. This is usually done without comment or mention, as though it is "obvious" we are still talking about the same objects.

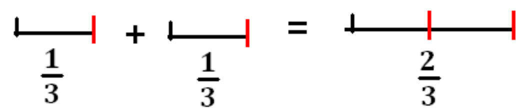
Model 3: *Fractions are points on the number line (and so are numbers in their own right).*

This is often motivated in a model 2 kind of way: *If I divide the unit interval into three equal parts, what pieces do I see?*



The location labeled $\frac{2}{3}$ is confusing.

We could go back to our model 1 thinking and add pieces of string-like pie:



But is $\frac{2}{3}$ a length of string or a point?

Maybe model 3 should be modified:

Model 3': *A fraction is a point on the number line, but it actually represents a length – namely the distance between it and the zero point.*

> **A Point of Confusion:**
>
> Okay. What about $\frac{7}{3}$?
>
> Is that $7 \times \frac{1}{3}$, the unit of $\frac{1}{3}$ added together seven times, or is it the result of dividing a length of seven into three equal parts? I am meant to believe these are the same. Is it obvious that they are?

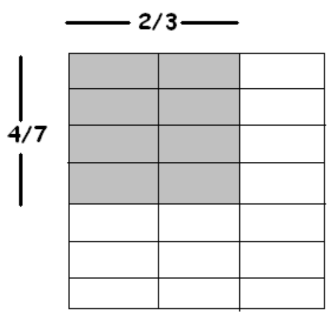
Assuming we can get past starting hick-ups …

In model $3'$ fractions are lengths and "multiplying lengths" has geometric meaning: we call that a computation of area. So let's use area to motivate the multiplication of fractions.

(Poor students who are still thinking pie from the previous year!)

Let's compute $\frac{4}{7}\times\frac{2}{3}$ as an area problem.

Start with a square and divide one side-length into sevenths and the other side-length into thirds and mark off the $\frac{4}{7}$ and $\frac{2}{3}$ positions. (Just as though the sides of the squares are unit lengths on the number line.)



The product $\frac{4}{7}\times\frac{2}{3}$ is the area of the shaded region shown.

But we see that the whole square is divided into 21 pieces in all and we've shaded 8 of them. This is $\frac{8}{21}$ of pie.

HANG ON! I thought we **weren't** using the pie model! (So we do want students to think of the pie model from last year at the same time?)

NO ONE MODEL DOES IT ALL

The reason why fractions are hard is because they are fundamentally an abstract concept. We can model them in different contexts, but no single model will capture all the features we feel hold true for these numbers called fractions.

Multiplication of fractions makes no sense in the concrete pie model 1. But it might make sense if we feel easy about a mix of models 3 and 1.

The addition of fractions is hard to understand in pie-per-boy model 2. (How do you think through $\frac{2}{5}+\frac{4}{3}$?).

But addition does make sense in model 3 – just add lengths of string together.

But model 3 has its limitations in adding fractions: *Okay, I can see what to do geometrically to find the point* $\frac{2}{5}+\frac{4}{3}$, *just lay one length after the other. But what do I do to actually compute its value?*

We present many different models of fractions to students throughout their school years, each with its limitations. And each time we say "This is what a fraction is." But then we abandon the model as soon as we hit a wall (and we always shall hit walls) and switch to a new model and say there: "This is what a fraction is."

Each model is like a blind man feeling an elephant. Each speaks a truth: "An elephant is a flat expanse of leather" (feeling its belly). "An elephant is a hard bone" (feeling the tusk). "An elephant is a length of rope" (feeling its tail). But no model actually says what an elephant is in totality. We've give students different aspects of a truth, but they feel contradictory, hazy at best.

WHAT IS THE TRUTH

As a mathematician I really don't know what a fraction actually is. From a study of abstract algebra I would just say that a fraction is a pair of integers a and b, with b non-zero, usually written above and below a vinculum, $\frac{a}{b}$, with two different expressions $\frac{a}{b}$ and $\frac{c}{d}$ deemed equivalent if $ad = bc$. (Mathematicians formally define what they mean by "equivalent" and what it means to have a whole class of objects deemed equivalent – an "equivalence class.") That is, mathematicians just use what they feel is intuitively true and turn that into the definition and thereby side-step the whole question as to what a fraction actually is!

SO WHAT TO DO FOR STUDENTS?

I am honest with my (middle- and high-school) students. I tell them that I don't know what a fraction actually is, and I invite a discussion about all the things they've seen and have been told about what fractions are. The pie model invariably comes up, as does a sharing model, and the idea of points on the number line. We honestly talk about how helpful/relevant each model is – up to a point – and how each model breaks down, just as I've described in this essay. We talk about whether or not these models are "obviously" equivalent – are they talking about the same things or different things? We wonder if each model is feeling a different part of an elephant. And we conclude in the end that we none of us really knows what a fraction actually is.

But during this discussion we do identify key features of fractions, as suggested by the various models, that seem intuitively right and natural.

For example, we all readily agree that the following two statements certainly feel right:

BELIEF 1: $\frac{a}{a} = 1$
for each positive whole number a.

BELIEF 2: $\frac{a}{1} = a$
for each positive whole number a.

From the basic pie model, adding fractions of like denominator feels natural and right too:

BELIEF 3: $\frac{a}{N} + \frac{b}{N} = \frac{a+b}{N}$
for whole numbers a, b, and N.

From pies per boy, we also have that the following feels natural and right. (It explains why $\frac{8}{10} = \frac{4}{5}$, for example, and so seems important.)

BELIEF 4: $\frac{ax}{bx} = \frac{a}{b}$
for positive whole numbers a, b and x.

At this point, we don't have any fundamental beliefs that mention multiplication. I press one more idea, one from the pies per girl model.

Suppose we share a pies among b girls. (Each girl currently gets $\frac{a}{b}$ pie per girl.)

How could we double the amount of pie each girl gets? <u>Answer</u>: Just double the amount of pie!

We have: $\frac{2a}{b} = 2 \times \frac{a}{b}$.

(This really is saying something: doubling the amount of pie doubles the amount of pie per girl.)

As there is nothing special about the number two here, this suggests:

BELIEF 5: $x \times \frac{a}{b} = \frac{xa}{b}$
for positive whole numbers a, b and x.

So the models we experience from grade-school do at least develop an intuitive base for these things we feel exist and are called fractions. And I've pulled out here five basic beliefs that feel particularly fundamental and right.

Continuing to be honest with my students I ask:

Do these beliefs feel so fundamental and so right that you think they should hold for <u>all</u> types of numbers – not just positive whole numbers? Do you want to play the game of exploring the full logical consequences of these beliefs and see where they take us?

I admit we still haven't said what a fraction is – we can't – but we've at least pinned down five pieces of their mathematical behavior.

And the beauty of these five basic properties is that all the remaining properties of fractions that feel familiar to us follow logically from them!

From beliefs 5, 4 and 2 we can prove, for example, that $7 \times \frac{3}{7}$, equals 3. (And in general that $b \times \frac{a}{b} = a$.)

We can use beliefs 4 and 3 to figure out the mathematics of adding fractions: to compute $\frac{3}{7} + \frac{6}{19}$, for example, as a consequence of these beliefs.

We can see how to use beliefs 5 and 4 lead us to multiplying fractions, to compute $\frac{3}{7} \times \frac{6}{19}$ in a snap, just as an mathematical consequence of the beliefs.

We can divide fractions using belief 4 and make the age old rule "to divide, multiply by the reciprocal" obvious and obsolete! (No need to ever say or even think it!)

We can explain why $\frac{-3}{8}$, $-\frac{3}{8}$ and $\frac{3}{-8}$ are all equivalent, using beliefs 4 and 5.

We can explain why $\frac{5}{0}$ has to be undefined and $\frac{0}{0}$ must be too for a different reason.

And so on!

Actually, everything we usually wonder about and want to do mathematically with fractions can be explained and justified as logical consequences of these five basic beliefs!

So, here's my story of fractions for students. During grades K-6 we do, of course, develop an intuitive understanding of fractions and through concrete models, "parts of wholes," and develop some of the mathematics that seems to be appropriate for those models. (Getting to the number of line and thinking in terms of "units of thirds" or "units of fifths," for example, is particularly helpful).

Then, sometime in grade 7 – 12, with a number of models in our minds from the past, we take a moment to reflect on our experiences and come to realize that, actually, it is not clear how all these models "hang" together. We let everything unravel in our reflections, and we explore the deficiencies of the models both individually and as a collective whole. Our job now is to be honest about fractions and admit that actually no one model captures everything we like to believe about them and how they behave.

This then begs the question: *So … what do we believe about fractions and how they should behave?*

We're then upfront about matters and just list our basic beliefs and call them what they are: beliefs!

And with the five particular basic beliefs I've listed pinned down, we marvel at the delight of seeing everything we were taught about the mathematics of fractions just unfold as a series of logical consequences of these basic beliefs.

And if you find a concrete model in which those five beliefs happen to apply - in thinking about lengths on a number line, or in thinking about sharing quantities - then all those mathematical consequences apply to that model too. (And we have to keep in mind that each model will be deficient, and so it will be hard, if not meaningless, to interpret some of the mathematical consequences within that model.)

SO … HOW DOES EVERYTHING FOLLOW FROM THOSE FIVE BASIC BELIEFS?

Rather than let this essay become a tome, let me refer you to:

www.jamestanton.com/?p=1461

for a piece that goes through all those details. Or better yet … Can you figure out for yourself the mathematical logic that explains all the ideas mentioned in the previous page?

I find that middle-school and high-school students enjoy this work. The honest admission that no one, including them, really is meant to "get" fractions from their early introductions to them is an incredible emotional relief. That fractions are abstract is the truth, and students can so handle the truth!

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