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TANTON'S TAKE ON …

NAPIER'S LOGARITHMS

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This material is based on content that has arisen in an experimental, free-form, question-and-response, email conversation/professional development experience currently underway with a group of 15 educators. It seems the historical story of logarithms is sorely neglected from the curriculum. Here are some questions and responses that have arisen from the discussion about the origin of logarithms.

These tidbits assume we already know the basic mathematics of logarithms. (See the April 2013 Curriculum Essay at www.jamesanton.com/?p=1072.)

WHO INVENTED LOGARITHMS AND WHY?

Response: During the Renaissance, science flourished in Europe. Scholars were collecting data and working with data to understand the world around them. And of course, they had to repeatedly perform arithmetic computations on large lists of data numbers to perform statistics, to analyze equations, and so on. But, of course, all this arithmetic had to be done with pencil and paper.

Now adding a large list of numbers is not fun, but it is doable. Multiplying a large list of numbers, on the other hand, is downright horrid.

3.17	3.17
+ 2.98	x 2.98
+ 3.02	x 3.02
+ 2.47	x 2.47
+ 3.28	x 3.28
=	=
Not fun, but doable	HORRID!

The fact that multiplying numbers is so hard and so tedious actually held back scientific progress all through the 1400s and 1500s!

So a Scottish mathematician by the name of John Napier (1550 – 1617) set out to ease the tremendous woe of all science and invent a method that would turn multiplication problems into addition problems.

Napier was an inventive and creative fellow. After much toying and playing, he came up with mighty complex method that did the trick. Napier imagined two particles each moving along a number line, one an infinite line and one a finite line. The first particle moved at a uniform speed and the second at a speed that varied according to the distance it still had to traverse across the finite line. (Mathematical historians, it seems, really are not sure what led Napier to consider this curious dynamic model.)

Napier found that comparing the distances traversed by each particle gave a means of computing values that could be used to turn multiplication problems into addition problems. He gave a name for his method based on the Greek word *logos* for ratio and *arithmos* for number, hence logarithm.

No one really understood Napier's approach. But the table of values he provided from it could be used with ease for arithmetic without regard to the theory that led to them.

Later, he and a colleague Henry Briggs (1561 – 1630), simplified the theory a tad and produced new tables of values – log tables – that were immensely popular.

By freeing up arithmetic, Napier's logarithms literally saved the progress of science.

value	logarithm
1	0
2	.301
3	.477
4	.602
5	.699
6	.778
7	.845
8	.903
9	.954
10	1

To compute 2x3

log(2) = .301
log(3) = .477

Add .778

We see this matches the answer 6.

It wasn't until another 150 years or so before scholars properly understood that Napier's logarithms were essentially exponents. But by then – and now another three hundred years later – the name *logarithm* for them was entrenched.

HOW DO YOU USE A LOG TABLE?

Response: Here's what one looks like.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1·0	·0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
1·1	·0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
1·2	·0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
1·3	·1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
1·4	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
1·5	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
1·6	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
1·7	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
1·8	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
1·9	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
2·0	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
2·1	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
2·2	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
2·3	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
2·4	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
2·5	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
2·6	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15

Each value in the table is a base-ten logarithm value. For example, we see in the table that $\log_{10}(1.40) = 0.1461$, $\log_{10}(1.41) = 0.1492$, and so on. Each answer is rounded to four decimal places.

This table actually offers you a bit more precision. To see the value of $\log_{10}(1.415)$, for instance, look at the number on the 1.4

row and under the digit 5 of the right section to see "15." This means we need to add the digits 1 and 5, in turn, to the final two digits of our answer to $\log(1.41)$.

$$\begin{aligned}\log_{10}(1.415) &= \log_{10}(1.41) + 0.0015 \\ &= 0.1492 + 0.0015 \\ &= 0.1507\end{aligned}.$$

But the point of these tables is that you don't need to know what any of these numbers actually mean! They simply let you compute multiplications.

For example, let's look up the answer to 1.41×1.22.

Now, according to the table 1.41 gives the number 0.1492 and 1.22 gives the number 0.0864. Add these.

$$0.1492 + 0.0864 = 0.2356$$

Looking at back at the table we see that 0.2355 appears as the table value for 1.72. This is close to 0.2356, so our product we seek must be close to 1.72. (The correct answer is 1.41 x 1.22 = 1.7202.)

Actually, on the table, we see that $\log_{10}(1.72) = 0.2355$ and $\log_{10}(1.721) = 0.2357$ so our answer is indeed 1.72 to two decimal places.

Although scientists did not know what the numbers in the table meant, they could nonetheless use the tables to compute products of numbers quickly. This freed up science in the 1600s and allowed scholars to work with data and scientific measurements with relative ease. Napier, and Briggs, provided a profound service to furthering scientific scholarship.

Comment: The log values of numbers below 10 are all "zero point something;" the log value of all numbers between 10 and 100 are all "one point something;" between 100 and 1000 all "two point something;" and so on. Log tables will only show the decimal part of log values, assuming you, the reader, already know what the integer part should be.

HOW DID NAPIER COMPUTE HIS LOG VALUES?

Response:
Napier toiled for many decades developing a technique that would provide a simple means for converting multiplication problems into addition problems. Today we understand his technique in terms of the powers of numbers: to multiply 10^a and 10^b, say, simply add the exponents. But this is not the approach Napier took.

Mathematical historians are not clear what lead Napier to work with a dynamic model—comparing the motions of two particles each on a straight line—and it took scholars of the time many decades to realise that Napier’s logarithms could be defined in terms of exponents. John Wallis in 1685 and then Johann Bernoulli in 1694 were the first to start seeing connections along these lines, but the mathematics of (the equivalent of) fractional exponents was not properly understood. It was not until the work of Swiss mathematician Leonhard Euler that a definitive theory of exponents was in hand and logarithms were finally seen for what they truly are. But by then, Napier’s name *logarithm* for these exponents-in-disguise was entrenched.

Computing Logarithms

Before we delve into the mathematics of Napier’s approach, let’s see how it is possible to compute some logarithmic values by hand with a clever choice of base. (We’ll use the modern notation of exponents here.)

Set $b = \left(1 - \frac{1}{10}\right)^{10}$.

Then

$$b^{0.1} = \left(1 - \frac{1}{10}\right)^{1} = 0.9$$

$$\begin{aligned} b^{0.2} &= b^{0.1} \times b^{0.1} = 0.9 \times \left(1 - \frac{1}{10}\right) \\ &= 0.9 - 0.09 = 0.81 \end{aligned}$$

$$\begin{aligned} b^{0.3} &= b^{0.2} \times b^{0.1} = 0.81 \times \left(1 - \frac{1}{10}\right) \\ &= 0.81 - 0.081 = 0.729 \end{aligned}$$

We see that each term is the previous value minus one tenth of the previous value. Thus $b^{0.1}, b^{0.2}, b^{0.3}, b^{0.4}, \ldots$ can all be readily computed by hand. We have

$$\begin{aligned} &\log_b(0.9) = 0.1 \\ &\log_b(0.81) = 0.2 \\ &\log_b(0.729) = 0.3 \\ &\ldots \end{aligned}$$

If we work with $b = \left(1 - \frac{1}{100}\right)^{100}$, we can readily work out $b^{0.01}$, $b^{0.02}$, $b^{0.03}$, … by hand, and thus some logarithm values to two-decimal places.

Napier, in his work, decided to work to seven decimal places and thus was working with the equivalent of a base of

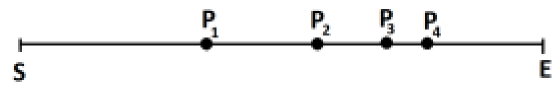
$$b = \left(1 - \frac{1}{10^7}\right)^{10^7}.$$

Towards Napier's Approach

Here is the gist of Napier's kinetic approach, developed after several decades of deep thinking, no doubt.

Napier envisioned a particle moving along a line segment, starting at one endpoint S and heading towards the other, E, and doing so in a way that its velocity decreased in a proportional way:

If, in equal time intervals, the particle moves from S to P_1 to P_2 and so on, then the proportions $\frac{SP_1}{SE} = \frac{P_1P_2}{P_1E} = \frac{P_2P_3}{P_2E} = \cdots$ are equal.



So if the particle moves one-sixth of the way in the first time period, say, then it moves one-sixth of the distance that remains in the next time period, one-sixth of what then remains the next time period, and so on. The velocity of the particle thus decreases with time and it approaches a velocity of zero as it nears E.

Suppose this particle starts with velocity v.

Then Napier imagined a second particle moving along an infinitely long line at constant speed v. This particle thus moves the same distance over equal time intervals.

Napier then, essentially, defined his logarithm of a number x as follows:

Compute the time t it takes for the particle reach the point P that is x units from E. Here

$$PE = x.$$

Then the Napier Logarithm of x is the distance, vt, the particle moving at constant velocity moves in this time period.

In order to compute times and distances in such a scenario, Napier set some convenient values. He set the line segment to be

$$SE = 10^7$$

units long, and chose an initial velocity of the particle so that it moves $\frac{1}{10^7}$ of the way each second. Thus

$$\frac{SP_1}{SE} = \frac{P_1P_2}{P_1E} = \frac{P_2P_3}{P_2E} = \cdots = \frac{1}{10^7}.$$

He was able to compute velocities and distances traveled for each whole number of seconds as follows.

From $\frac{P_nP_{n+1}}{P_nE} = \frac{1}{10^7}$ we get

$$\frac{P_nE - P_{n+1}E}{P_nE} = \frac{1}{10^7}$$

giving

$$P_{n+1}E = \left(1 - \frac{1}{10^7}\right)P_nE.$$

Starting with $SE = 10^7$ we thus get

$$P_1E = \left(1 - \frac{1}{10^7}\right)SE = \left(1 - \frac{1}{10^7}\right)10^7$$

and each value P_nE is the previous value minus $1/10^7$ of that value. Thus, in N seconds, the particle is at a position that is $P_NE = \left(1 - \frac{1}{10^7}\right)^N \times 10^7$ units away from the endpoint E, So, according to Napier's definition, if $x = \left(1 - \frac{1}{10^7}\right)^N \times 10^7$, then

$$NapierLog(x) = N.$$

As the previous section showed, we can compute such values with ease. This thus gave Napier a whole bank of log values to work with which he could then use interpolation and other mathematical techniques to estimate intermediate logarithm values. (This was by no means easy work and the inaccuracies that arose from estimations caused problems for scientists attempting to do higher and higher precision calculations. Many scholars throughout the 1600s worked hard to create more and more accurate tables of log values.)

In modern notation, we see from $x = \left(1 - \frac{1}{10^7}\right)^N \times 10^7$ that

$$NapierLog(x) = \log_{1-\frac{1}{10^7}}\left(\frac{x}{10^7}\right).$$

Finishing Napier's Approach

At Napier's time astronomers were using extensive tables of sine and cosine values and many of their calculations involved multiplying and dividing these trigonometric values. To address the concerns of astronomers, in particular, Napier assumed that his number x, the distance of a point x from the endpoint along the segment SE, was actually a trigonometric value $\sin(\theta)$ for some angle θ, usually multiplied by a large power of ten to help avoid decimals, say,

$$x = 10^9 \sin(\theta).$$

So Napier actually gave tables of "logarithms of angles," which, in modern notation, would be equivalent to something like

$$NapierLog(\theta) = \log_{1-\frac{1}{10^7}}\left(\frac{10^9 \sin(\theta)}{10^7}\right).$$

To actually see that Napier's approach is equivalent to computing a logarithm as we know it, one must use the techniques of calculus. These techniques were not available to Napier at the time, but he developed intuitive arguments, which turned out to be correct, to justify some of the assumptions he made about his moving particles.

For instance, Napier argued that a particle that moved equal proportions of distances remaining along the line segment SE during equal time intervals must actually have the property that its velocity at any point P is proportional to its remaining distance from the endpoint E. In calculus this allows us to write and solve a differential equation which shows that the particle at any time t is at a distance Ae^{kt} units from endpoint E for some constants A and k. Computing the time taken to reach a particular position along the line, that is, given the value Ae^{kt} and computing t from it, is equivalent to computing a logarithm.

FOR MORE:
See http://www.maa.org/press/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-john-napier-introduces-logarithms .

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