CURRICULUM INSPIRATIONS: www.maa.org/ci

MATH FOR AMERICA_DC: www.mathforamerica.org/dc

INNOVATIVE CURRICULUM ONLINE EXPERIENCES: www.gdaymath.com

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TANTON'S TAKE ON …

SAME QUESTIONS

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Here are two little questions:

1. *Twelve white dots sit in a row. In how many ways can I color two red?*

2. *How many solutions does the equation* $10 = x + y + z$ *possess with* x, y, *and* z *each a non-negative integer?*

These two very questions regularly make their way into one of my permutations and combinations sessions for educators, or for students. Participants recognize the answer to the first question as $\frac{12!}{10!2!} = \frac{12\times 11}{2} = 66$. The second question, however, is usually seen as jarring: an algebra question in a class on counting?

But these two questions are just the same one question in two different guises. I've been trained as a mathematician to always be on the lookout for different costumes enshrouding identical ideas—but I wasn't as a school student. I worry that this is still the case for school students today.

Here's how to connect the two questions:

Each picture of two red dots among twelve has a certain count of white dots to the left of the reds, x of them, say, a certain count of white dots between the two red dots, y of them, and a certain count of white dots to their right, z of them. As there are ten white dots in all, $x + y + z = 10$.

$2 + 5 + 3 = 10$

Each picture can thus be interpreted as a solution to the algebra equation and, conversely, each algebra solution can be encoded as a picture.

Thus the number of solutions matches the number of possible pictures, 66 of them.

Exercise: *How many non-negative integer solutions does the equation* $100 = a+b+c+d$ *possess?*

As soon as one sees how to reinterpret one question, one can't help but want to do it some more. Here are four more questions, each necessarily with answer 66 as each is really the same question again. (Can you see how? Can you add a few more questions to the list?)

3. *Twelve dots are drawn on a circle. If I draw a chord between each and every pair of dots, how many chords will I draw in total?*

4. *A dozen people meet in a room. If each person shakes hands with each and every other person exactly once, how many handshakes take place in total?*

5. *An ice-cream stand is offering a "mega-bowl" special: ten scoops of ice-cream from three different choices of flavors. Assuming scoops can jumble about in the bowl, how many different mega-bowl specials are there?*

6. *How many solutions are there to the equation* $13 = x+y+z$ *with* x, y, *and* z *each a positive integer?*

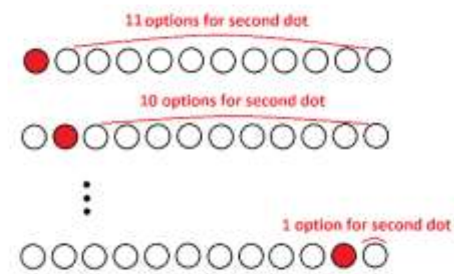
Mulling on alternative formulations and approaches is fun. (Mathematics is not a series of tasks to be solved. Mathematics is a series of tasks to be mulled over, reinterpreted, and revisited over and over again!)

Here's a philosophical seventh question that comes from mulling on the counting process of question 1.

7. *How can we use the two red dots among twelve to see, without actually doing the arithmetic, that the sum* $1+2+3+4+5+6+7+8+9+10+11$ *must equal* 66 *?*

Here are my thoughts:

We want to count all possible ways to color two dots red. Let's consider all options for coloring a left dot and see how many choices each scenario leaves for coloring a second dot.



The picture shows that that there are $1+2+3+\cdots+11$ ways to color two dots. Thus sum must thus be 66.

In general, we realise that

$$1+2+3+\cdots+n = {}_{n+1}C_2 = \frac{n(n+1)}{2}.$$

Exercise: *a) Use three red dots to show that*

$$\begin{array}{l} 1+ \\ 1+2+ \\ 1+2+3+ \\ \vdots \\ 1+2+3+\cdots+n \end{array} = {}_{n+2}C_3$$

b) What is the value of $n\times 1+(n-1)\times 2+\cdots+1\times n$ *?*

Exercise: *For twelve dots drawn on a circle I draw a chord between each and every pair of dots. The dots are non-symmetrically spaced so that only two chords ever pass through a point of intersection. Use a picture to explain why there are* ${}_{12}C_4$ *intersection points.*

For a real challenge… n *dots are drawn on a circle and chords are drawn between each and every pair of dots. The dots are non-symmetrically arranged so that no more than two chords ever pass through an intersection point. Explain why the number of regions in the diagram equals*

$1 + {}_nC_2 + {}_nC_4$.

(See www.jamestanton.com/?p=775 .)

By the way, the sequence of numbers that arise from this formula begins $1, 2, 4, 8, 16$ and then continues $31, 57, 99, \ldots$.

PROBLEMS THAT ARE THE SAME IN THE CURRICULUM

To me permutation and combination puzzles are just the same problem over and over again in disguise. (Why do we make this topic so hard?) For example, to count how many ways to select three people from six, just count how many ways to arrange the letters CCCXXX. (C = chosen and X = not chosen). To count how many ways to select the three people in order, just count the number of ways to arrange the letters FSTXXX. (First, second, third, not chosen.) To learn how easy it is to count letter arrangements, see www.gdaymath.com/courses/permutations-and-combinations/ .

The Common Core State Standards (the A.SSE standards) want students to see structure in expressions, that factoring $x^4 - y^4$, for example, starts off as a standard problem in disguise (to factor $x^2 - y^2$).

Solving $(x+3)^2 = 25$ is a breeze: it simply reads as "something squared is twenty-five." Solving $x^2 + 6x + 9 = 25$ is just the same question in disguise. This represents the best way to solve quadratics, to simply recognize each equation as something easy in costume! See details at www.gdaymath.com/courses/quadratics/ .)

Computing $276 \div 12$ in grade 5 is identical to computing $(2x^2 + 7x + 6) \div (x+2)$ in grade 10. It is the same work in different guises – and Exploding Dots makes this blatantly clear. (See www.gdaymath.com/courses/exploding-dots/ .)

Young students have little trouble graphing scatter plots (data points of students' shoe size and height, for example). It is natural and conceptually easy. Yet graphing equations and functions, in general, is unnatural and hard for many students. Why then don't we have students just plot scatter plots all through middle school and refer to all graphing thereafter as scatter plots in different guises: the graph of $x^2 + y^2 = 1$ is the scatter plot of all data points that make this equation true; the graph of a function is the scatter plot of all input/output data pairs. Actually, this is exactly what I am about to advise a new curriculum to do!

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