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TANTON'S TAKE ON …

"SIMPLIFY"

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"Professionalism" in mathematics education has adopted a strange interpretation over the past century. To be an effective educator it seems that we must be technically precise at all times, be the expert in front of the classroom at all times, and obviate all possible points of confusion for the students before any element of confusion could arise. Educators and curriculum writers should break down complex ideas into gradual manageable pieces, carefully delivering the pieces one at a time, so that students are well prepared for when the big picture idea eventually arises. Techniques are taught in earlier grades priming students for work in the later grades. ("We do this now because they will need to know it later on.") Jargon needs to be clear and precisely defined and used absolutely grammatically correctly right from the get-go. Ideas are to be formalized and procedures are to be agreed upon before entering the classroom to ensure uniformity, clarity, and precision. In short, professionalism in mathematics teaching, interpreted this way utterly fails to do what we mathematics educators need to do for students, that is, to actually model the learning process!

For best practice we need to not know answers and show our students how to figure things out. We need to allow for confusion and teach the art of clarifying our thinking and working one's way out of haziness. We need to develop working definitions and learn what to do when we discover they are not adequate. We need to take complex problems and teach how to break them down into manageable pieces. We need to teach the art of communicating ideas adequately, if not beautifully. We need to teach the power of common sense!

The result of "expert teaching practices" over the past decades is a standard curriculum that, over and over again, lacks context and purpose for the tasks presented. The Common Core State Standards, whatever you might think of them, are working to turn that around. The eight Standards for Mathematical Practice, in my opinion the backbone of the Common Core, are all about teaching meaningful thinking. Even states that have rejected the Common Core have, or are working on, curricula that put thinking and doing-with-context into the foreground of mathematical classroom practice.

ON "SIMPLIFY"

I have been speaking up against the use of the word "simplify" for over a decade. Without context, the command actually means nothing. I am absolutely delighted to see that the Common Core has called for the removal of the blanket use of this as a textbook command too. Hooray!

The flagrant use of the word *simplify* comes from having broken up the student learning experience into tinier and tinier, allegedly digestible, micro-pieces to the point that the original context and purpose of those pieces are lost. They have become procedural tasks for rote doing, nothing more.

To explain the issue, consider a classic algebra textbook problem:

Exercise: Simplify $\sqrt{20}$.

Answer: *It looks great as it is!*
(And I actually mean this: $\sqrt{20}$ really does look visually cleaner to me than $2\sqrt{5}$.)

A really intelligent response is the following: *What I do with $\sqrt{20}$ depends on what comes next! What's the task we're wondering about that involves $\sqrt{20}$?*

If I am wondering whether $\sqrt{20}+\sqrt{5}$ is the same as $\sqrt{25}$, for example, then I might write $\sqrt{20}+\sqrt{5}=2\sqrt{5}+\sqrt{5}=3\sqrt{5}$, to see this is not 5. (We have: $3\sqrt{5}>3\sqrt{4}=6$. It is definitely not 5!)

If, on the other hand, $\sqrt{20}$ is part of the calculation $\sqrt{20}+2\sqrt{20}$, then I'd leave $\sqrt{20}$ as it is and write the obvious answer $3\sqrt{20}$.

Without a context, there is really nothing to do.

Comment: I am fully aware that textbook authors given the following definition: *A radical $\sqrt{N}$ is deemed to be* <u>*simplified*</u> *if it is rewritten in the form $a\sqrt{b}$ where b is an integer possessing no square factors other than 1.* But I am yet to see an explanation as to *why* one generally wants to rewrite radicals this way in a knee jerk fashion.

Don't get me wrong: In my algebra classes I did teach the technique of rewriting radicals in different ways. This helped us develop flexibility for handling different tasks and problems. But we used the problems to motivate the need for a mathematical technique and not have the technique just be another item to "get through" in a textbook list of things called *algebra*.

Question: Do two squares, one of area 40 and one of area 80, fit inside a square of area 160?

I've also noticed in the last six years or so some easing of the obsession to "rationalize radicals in the denominator." There was a context for this that was relevant and meaningful all through the 1800s and the years up to about 1980. It has only taken 30 years for the mathematics curriculum world to realize that there is no need for this anymore.

Exercise: Simplify $\frac{1}{\sqrt{2}}$.

Mathematician's answer: *There is nothing mathematically objectionable to the expression as it is. We can leave it alone!*

Now, if I am an engineer in the 1930s in the process of designing a bridge, say, and I need to know the value of $\sin\left(45^{\circ}\right)=\frac{1}{\sqrt{2}}$ to three decimal places, then I have an issue. I would go to my booklet of table values (handed out to every school kid up through the 1970s), read that $\sqrt{2}\approx 1.414$, and balk at the idea of performing an unfriendly long division by hand:

$$1.414\,\overline{)\,1.0000000\cdots}$$

Not fun!

In this context, I might then choose to follow the general advice of school teachers at that time:

If you ever have a radical in a denominator of an expression and you actually need its decimal approximation, try multiplying the numerator and denominator of the rational expression each by that radical. It will make the paper-and-pencil calculations more manageable.

Rewriting $\frac{1}{\sqrt{2}}$ as $\frac{\sqrt{2}}{2}$ does indeed make the paper-and-pencil work considerably easier. We can see:

$$\frac{\sqrt{2}}{2}\approx\frac{1.414}{2}=0.707\,.$$

Today's context? If I need to know the three-decimal approximation of $\frac{1}{\sqrt{2}}$ there is no need to rationalize the denominator! Just type in $1/\mathrm{sqrt}(2)$ into the calculator and be done! (But do feel free to type in $\mathrm{sqrt}(2)/2$ if you prefer.)

Again, don't get me wrong. My algebra students became versatile in both rationalizing denominators and rationalizing numerators! The point was to learn how to manipulate the appearance of a radical in an expression to a place that is more appropriate for the problem at hand. There is no such thing as "simplifying" a stand-alone radical quantity.

Comment: It is appropriate to ask students to practice algebraic manipulations for the sake of developing fluency. Perhaps phrase questions this way:

$$\textit{Make } \frac{x^2\sqrt{xy^3}}{\sqrt{\frac{y^2}{x}}} \textit{ look friendlier.}$$

and realize that the answer here is subjective. That's okay!

Or you can be more precise:

Show that $\frac{x^2\sqrt{xy^3}}{\sqrt{\frac{y^2}{x}}}$ *can be rewritten in the form* $x^a y^b$ *for some real numbers* a *and* b. *What is the value of* $a+b$?

But don't let these workout questions become the defining experience of algebra class! Make sure the message is clear and obvious that we're solving problems, and that we're just pausing <u>momentarily</u> to strengthening these algebra muscles so that we can continue figuring out clever ways to solve meaningful, complex problems. Context is key, and the context comes from interesting queries first and foremost.

AP CALCULUS:

There is one place in the standard curriculum where teachers must explicitly instruct students not to simplify. It occurs in the topic of implicit differentiation in AP Calculus. Consider, for example, the question:

What is the slope of the tangent line to the algebraic curve given by $xy + x^2 + y^2 = 3$ *at the point* $(1,1)$ *?*

Assuming that y can, in fact, be regarded as a function of x near the point $(1,1)$ we can differentiate this expression with respect to x obtain:

$$y + x\frac{dy}{dx} + 2x + 2y\frac{dy}{dx} = 0 .$$

This gives:

$$\frac{dy}{dx} = -\frac{y+2x}{x+2y} .$$

At the point $x = 1,\ y = 1$ we see $\frac{dy}{dx} = -1$.

Alternatively, to save twenty seconds of precious time, don't solve for $\frac{dy}{dx}$. Just put $x = 1,\ y = 1$ into:

$$y + x\frac{dy}{dx} + 2x + 2y\frac{dy}{dx} = 0$$

to get $1 + \frac{dy}{dx} + 2 + 2\frac{dy}{dx} = 0$ giving $\frac{dy}{dx} = -1$ at this point.

Question: What is the best approach for this task: *Find the slope of the tangent line to the curve* $(xy)^2 = y^x$ *at the point* $(1,1)$.

In fact, AP calculus teachers often advise:

Don't ever simplify the expression one obtains after implicit differentiation: Just plug in the given x *and* y *values and go from there!*

Calculus teachers are often surprised at how hard it is to break students out of the simplifying habit. This would be a non-issue if we encouraged students from the get-go to look at tasks for what they are and assess for themselves the most common-sense, appropriate method for each in turn. Eleven years of "simplifying" for the sake of rewriting expressions in some pre-described form irrelevant of context does not serve students well.

Let's do indeed delete the word *simplify* from our teaching!

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