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TANTON'S TAKE ON …

SYMMETRY THROUGHOUT MATHEMATICS

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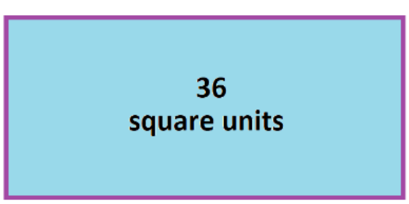
Symmetry is a powerful friend in mathematics. We should invite her in whenever we suspect she might be lurking nearby.

My standard illustration of her power is the following: *I have a rectangle of area 36 square units. What can you deduce about my rectangle? Answer: Nothing! You can't guess what its dimensions might be:* 4×9*?* $4\frac{1}{2}\times 8$*?* $\sqrt{6}\times 6\sqrt{6}$ *? You can't know.*



But suppose I add that my rectangle is symmetrical. Then you know everything about my figure: it must be a 6×6 *square!*

Using the symmetry of a square makes solving quadratics a breeze! (See part 2 of http://gdaymath.com/courses/quadratics/.)

And mathematics itself has a penchant for symmetrical rectangles: Of all the rectangles of area 36 square units, the square has the smallest perimeter. (If soap bubbles were rectangles, they'd all settle to being squares.) And of all rectangles with a fixed perimeter, the square encloses the largest area, an observation we explored in this recent essay: http://www.jamestanton.com/wp-content/uploads/2012/03/Teaching-Problem-Solving-Mindset-_-10-Reflecting.pdf.

Word and Symbol Symmetry

Symmetry need not appear as geometric symmetry. Consider, for example, the following problem.

I will roll a die repeatedly. What are the chances I will see a 1 before I first roll a 6?

Failing to roll a 1 before rolling a 6 means to roll a 6 before first seeing a 1. The complement of what we seek is philosophically identical to the original task at hand; the labels "1" and "6" are simply switched.

Since the task and its complement are the same challenge, they must each have 50% chance of occurring. Done!

Thanks to symmetry we can also see immediately that the equation

$$(a+b)^3 = a^3 + b^3 + 3a^2b$$

cannot be correct: the left side is an expression that remains the same if we switch the roles of a and b, but the expression on the right does not.

This idea of symmetry in words and symbols is lovely. The branch of Projective Geometry (in which we declare parallel lines *do* meet at a point, a special "point at infinity") is symmetrical with its use of the words *point* and *line*: switching these words in any statement proved true in this geometry automatically leads to another statement which must also be true the in the geometry. It feels like magic!

And here's a curious property of the prime numbers.

Write out a list of primes, skipping some if you like, repeating others multiple times if you like.

$$P: \ 2,2,3,7,7,11,13,13,13,13,17,....$$

Count how many of these number are less one, how many are less than two, how many are less than three, and so on. This produces the "frequency sequence" of your list.

$$\begin{aligned} P: &\quad 2,2,3,7,7,11,13,13,13,13,17,.... \\ P^*: &\quad 0,0,2,3,3,3,3,5,5,5,5,6,6,10,10,.... \end{aligned}$$

Now compute the frequency sequence of your frequency sequence: the number of entries that are less than one, less than two, and so on.

$$\begin{aligned} P: &\quad 2,2,3,7,7,11,13,13,13,13,17,.... \\ P^*: &\quad 0,0,2,3,3,3,3,5,5,5,5,6,6,10,10,.... \\ P^{**}: &\quad 2,2,3,7,7,11,13,13,13,13,.... \end{aligned}$$

You are guaranteed to return to the list you started with!

Symmetry explains what is going on.

Represent the terms of the original sequence with dashes, drawing dots to their left to indicate their values. The first dash in the picture below has 2 dots to its left, the second dash has 2 dots to its left, the third dash has 3 dots to its left, the fourth dash has 7 dots to its left, and so on.

So here:

$P_n =$ the number of <u>dots</u> to the left of the *n*th <u>dash</u>.

The frequency sequence counts how many terms in the original sequence are smaller than a given value. For instance, $P_{10}{}^*$ is the number of dashes in our picture with less than 10 dots to their left. This would be all the dashes to the left of the 10th dot. That is, $P_{10}*$ is the count of dashes to the left of the 10th dot.

$P_n* =$ the number of <u>dashes</u> to the left of the *n*th <u>dot</u>.

The definitions of a sequence and its frequency sequence are identical except for an interchange of the words "dot" and "dash." Thus if we take the frequency sequence of a frequency sequence (that is, interchange these words twice), we must be back to the original sequence!

(By the way: There is nothing special about the prime numbers here: any non-negative, non-decreasing sequence of integers possesses this property!)

A Surprising Appearance of Symmetry

In the world of counting numbers, multiplication is often introduced as repeated addition through "groups of" thinking.

$$\begin{aligned} 4\times 5 &= \text{four groups of } 5 \\ &= 5+5+5+5 \\ &= 20 \end{aligned}$$

This definition is asymmetrical! The number four here is being used as some kind of "operator" or "call for action" and the number 5 is the object being acted on. This imbalance is more transparent if we think of concrete units:

four groups of 5 apples.

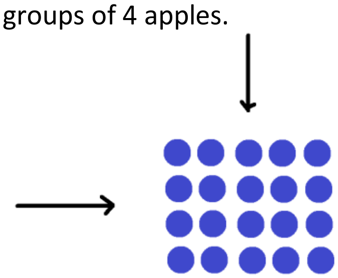
Here 5 comes in units of apples but the number four does not. (If "4" and "5" are both in units of apples, then 4×5 should equal "20 apples squared." Does that make sense?)

Some might argue that the number four here is unitless, or perhaps has units of "groups." But we do see that it is qualitatively different from "5 apples."

So it is should be utterly surprising then that 5×4 happens to have the same final answer of 20 apples, and that $a\times b$ is sure to give the same numerical answer as $b\times a$ for all possible counting numbers a and b.

But we do believe that multiplication is commutative!

Often this symmetry of arithmetic is justified via a picture that shows how to look at a collection of 20 apples two ways, both as four groups of 5 apples and as five groups of 4 apples.



Moreover, despite this picture being an illustration of literally 20 apples, it has a clear universal message: a groups of b apples will always be able to viewed as b groups of a apples.

Here the symmetry of arithmetic is being illustrated by two viewpoints of one asymmetrical diagram. (That's a little curious in its own right.)

Pushing for Symmetry

Pushing for symmetry in asymmetrical situations can lead to new insights. My favorite example of this is the standard algebraic argument "proving" that $0.9999\cdots$ equals 1.

Let

$$x = 0.9999\cdots.$$

Then

$$10x = 9.9999\cdots$$
$$= 9 + 0.9999\cdots$$
$$= 9 + x$$

giving $x = 1$.

But why this bias for infinitely many nines to the right of the decimal point? What about infinitely many nines to its left? What is the value of $\cdots 9999$?

Following exactly the same algebra …

Let

$$w = \cdots 9999 .$$

Then

$$10w = \cdots 9990$$
$$= \cdots 9999 - 9$$
$$= w - 9$$

qiving

$$w = -1.$$

The conclusion here does one of two things for us. It either puts into doubt our first argument—I don't believe the algebra in this second argument, so why should I believe the algebra in the first? Maybe I can't say that $0.9999\cdots$ is 1 after all?—or it leads us to a curious result about the infinite sum $9 + 90 + 900 + 9000 + \cdots$. Maybe there is an unusual system of arithmetic for which it is meaningful to examine such a sum and in that system it does, somehow, have value -1?

Entertaining this second idea opens our minds to a new world of wonderful mathematics motivated by a push for symmetry. (See http://gdaymath.com/lessons/explodingdots/7-3-optional-believe-infinite-sums/ for instance.)

Students exploring *Exploding Dots* (www.explodingdots.org) naturally call for symmetry in this very same way. They usually ask: "We have all these boxes going to the left. What about making boxes to the right as well?" They discover new numbers called decimals.

The typical mathematics curriculum is ripe with opportunity for the deep play of mathematical ideas. Symmetry is one prime example of such story sitting behind the scenes ready to provide insight and surprise. Let's invite her to come play whenever we can!

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