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TANTON'S TAKE ON …

WHAT IS A VECTOR?

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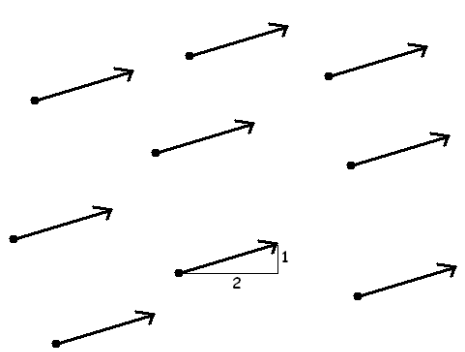
See the chapter (and videos) there for full details on this work.

A *vector* is just a shift, that's all. (I never really got this "a quantity with both magnitude and direction" thing.)

For example, suppose an earthquake hits town and shifts all points in town two units east and one unit north. We can draw arrows to indicate the translates of points.



The shift indicated here is an example of a vector. It has a horizontal component of 2 and a vertical component of 1 and is denoted as a pair of these numbers. It has become the convention to use angled brackets for vectors. (Round brackets makes us think of points.)

$$\text{vector} = < 2, 1 >$$

Comment: Although a vector is depicted as an arrow drawn on a page, the location of that arrow is immaterial. A vector represents a shift of *all* points.

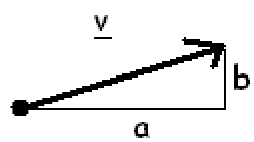
Vectors are often denoted by letters written in bold or with an underscore. For example, if we call the above shift "v" then we write:

$$\mathbf{v} = < 2,1 > \text{ or } \underline{v} = < 2,1 > .$$

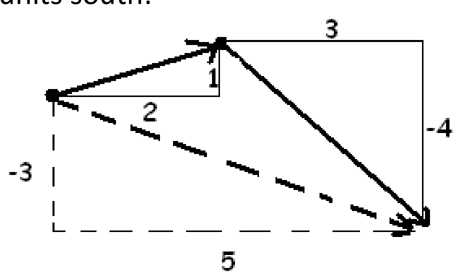
If one earthquake shifted points further than did another, then it seems reasonable to say that the first earthquake was stronger. The distance points are shifted is a measure of the strength of the quake.

The *magnitude* of a vector $\underline{v} = < a, b >$ is the distance each point moves during the shift given by the vector. It is denoted $\|\underline{v}\|$. Geometrically the magnitude of a vector is the length of any arrow used to depict it and is calculated via Pythagoras's theorem:

$$\| \underline{v} \| = \sqrt{a^2 + b^2}$$



ADDITION OF VECTORS:
If an earthquake that shifts all points 2 units east and 1 unit north is followed by a second earthquake that shifts all points 3 units east and 4 units south, the net result is a shift 5 units east and 3 units south:



We have:

$$< 2,1 > + < 3,-4 > = < 5,-3 > .$$

This suggests defining addition of vectors by the rule:

If $\underline{v} = < v_1, v_2 >$ and $\underline{w} = < w_1, w_2 >$, then $\underline{v} + \underline{w} = < v_1 + w_1, v_2 + w_2 >$.

Geometrically, this is performed by adjoining the arrow representing the second vector to the endpoint of the arrow representing the first vector, and identifying the resultant shift.

Exercise: Algebraically it is obvious that the addition of vectors is commutative: $\underline{v} + \underline{w} = \underline{w} + \underline{v}$. Is this obvious geometrically?

Exercise: If $\underline{v}$ is a vector, what meaning should we give to $2\underline{v}$? To $\frac{2}{3}\underline{v}$? To $-\underline{v}$? Answer this question both algebraically and geometrically.

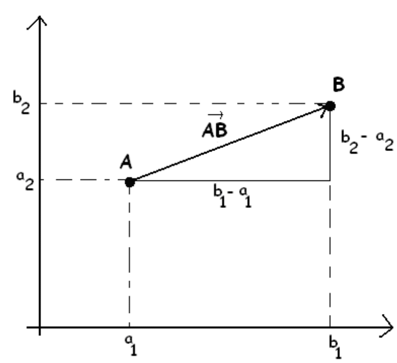
Exercise: The vector $\underline{0} = < 0,0 >$ is called the zero vector. Describe its geometric effect.

Exercise: If $\underline{v}$ is a vector, and c is a positive real numbers, what can you say about the magnitude of $c\underline{v}$ compared to the magnitude of $\underline{v}$? What if c is negative?

Exercise: If $\underline{v}$ and $\underline{w}$ are vectors, what is the geometric meaning of $\underline{v} - \underline{w}$? (Does it help to think of this as the addition of $\underline{v}$ and $-\underline{w}$?)

VECTORS BETWEEN POINTS

Two points A and B in the plane naturally define a vector, namely, the shift required to take A to B. This vector is denoted $\overrightarrow{AB}$.



If the two points have coordinates $A=(a_1,a_2)$ and $B=(b_1,b_2)$, then the required shift from A to B is:

$$\overrightarrow{AB} = < b_1-a_1, b_2-a_2 >$$

Some people will write this as:

$$\overrightarrow{AB} = B - A$$

as the difference of two points.

Technically, one cannot subtract points, but the mathematics implied by such a statement at the component level is correct! The interpretation at this component level is:

$B-A$ is a quantity with horizontal component the difference of the horizontal component of A and B, and vertical component the difference of the vertical components of A and B. These are indeed the components of the shift $\overrightarrow{AB}$.

Example: If $A=(-3,4)$ and $B=(9,2)$ then $\overrightarrow{AB} = <12,-2>$.

ASIDE: Many textbooks authors frown upon this approach and insist that only vectors can be added and subtracted. They will go to some pains to distinguish between a point A in the plane and the vector $\overrightarrow{OA}$ that represents the shift from the origin to A. We then have the statement:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

which is the technically correct vectorial interpretation of $\overrightarrow{AB} = B - A$.

Exercise: Let $R=(-3,7)$, $S=(0,-16)$ and $T=(8,3)$.

a) Compute $\overrightarrow{RS}$ and $\overrightarrow{SR}$ and show that $\overrightarrow{RS}+\overrightarrow{SR}$ is the zero vector. Does this make sense geometrically?

b) Draw a quick sketch of the three points R, S, and T on the plane and use it to show that $\overrightarrow{RS}+\overrightarrow{ST}+\overrightarrow{TR}$ should be the zero vector. Verify that it is by computing $\overrightarrow{RS}$, $\overrightarrow{ST}$ and $\overrightarrow{TR}$ and their sum.

Comment: We have $-\overrightarrow{AB} = \overrightarrow{BA}$.
(This is obvious in our illegal notation too: $-\left(B-A\right) = A-B$!)

VECTORS AND GEOMETRY

One can make exceptionally good use of our naughty "arithmetic of points" context, all true at the component level. Notice, for instance:

Given two points A and B in the plane, one can reach the point B by starting at A and then moving from A to B. Thus we can write:

$$B = A + \overrightarrow{AB}.$$

And this is correct:

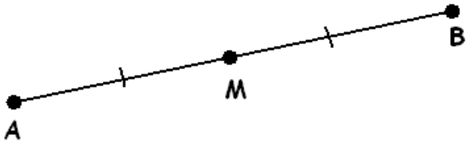
$$A + \overrightarrow{AB} = A + \left(B-A\right) = B.$$

Comment: This will still make many a curriculum writer cringe, but, as I say, it is valid at the component level:
"$A+\overrightarrow{AB} = A+(B-A)$" has x-coordinate $a_1+\left(b_1-a_1\right)=b_1$ and y-coordinate $a_2+\left(b_2-a_2\right)=b_2$.

Actually it is valid at the level of points too - if you choose to view complex numbers as points in the plane. Then adding and subtracting the points is legitimate mathematics for sure. (See the notes cited at the beginning of this essay for more.)

A Nice Example: If one starts at A and walks half way along $\overrightarrow{AB}$ we should reach the midpoint M of $\overline{AB}$:

$$M = A + \frac{1}{2}\overrightarrow{AB}.$$



Notice that

$A+\frac{1}{2}\overline{AB}=A+\frac{1}{2}(B-A)=\frac{A+B}{2}$ and so the midpoint of the line segment can be computed as: $M=\frac{A+B}{2}$.

For example, if $A=(3,4)$ and $B=(-1,8)$, then $M=\left(\frac{3+(-1)}{2},\frac{4+8}{2}\right)=(1,6)$.

Question: Do you see the formula $M=\frac{A+B}{2}$ working at the component level as the standard midpoint formula?

EVEN BETTER!

Let $A=(20,-30)$ and $B=(-5,15)$.

a) Find the coordinates of a point R that is one-third of the way along the line segment $\overline{AB}$, closer to A than to B.

b) Use the distance formula to find the distance between A and B, and between A and R. Check that AR really is one third of AB.

Let's go up some dimensions!

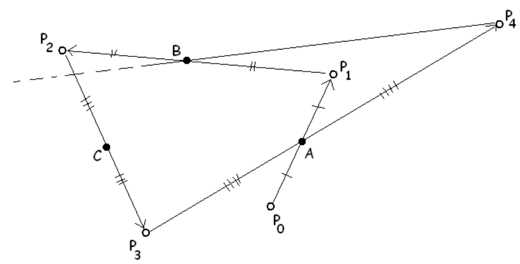
c) Let $A=(2,4,0)$ and $B=(7,-2,6)$ be points in three-dimensional space. Find the coordinates of a point one third of the way along the line segment $\overline{AB}$, closer to A than to B.

d) Let $A=(2,3,1,5,6)$ and $B=(-1,-2,-2,2,1)$ be two points in five-dimensional space. Find the coordinates of a point one third of the way along the line segment $\overline{AB}$, closer to A than to B.

FUN APPLICATIONS:

LEAP FROG

Three points A, B, and C are situated in the plane. A frog, starting at a position P_0, jumps in a straight line towards, and over the point A to land at a position P_1 with distances P_0A and AP_1 matching.



The frog then jumps in a straight line towards and over B to land at position P_2 with distances P_1B and BP_2 matching. And then again towards point C, and then towards point A, then B, then C, then A, then B, and so on, to create a sequence of landing positions $P_3, P_4, \ldots$ with points A, B, C cycling as midpoints of the consecutive segments $\overline{P_iP_{i+1}}$.

Does the frog ever return to start?

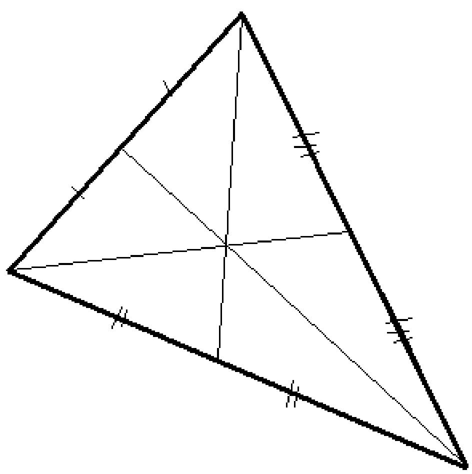
Play with this puzzle for a while. Try mapping out the path of a frog on a blank piece of paper using a ruler to measure distances fairly accurately, or better yet, on graph paper staying on intersection points. What amazing thing do you notice?

Answer: By our work in vector theory we have:

$$P_1 = P_0 + 2\overrightarrow{P_0A} = 2A - P_0$$
$$P_2 = P_1 + 2\overrightarrow{P_1B} = 2B - P_1 = 2B - 2A + P_0$$
$$P_3 = P_2 + 2\overrightarrow{P_2C} = 2C - P_2 = 2C - 2B + 2A - P_0$$
$$P_4 = P_3 + 2\overrightarrow{P_3A} = 2A - P_3 = 2B - 2C + P_0$$
$$P_5 = P_4 + 2\overrightarrow{P_4B} = 2B - P_4 = 2C - P_0$$
$$P_6 = P_5 + 2\overrightarrow{P_4C} = 2C - P_5 = P_0$$

After six jumps the frog is guaranteed to return to start! □

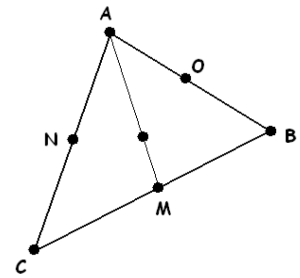
EXAMPLE: A *median* of a triangle is a line from one vertex of the triangle to the midpoint of the opposite side.



Prove that the three medians of a triangle are concurrent, that is, pass through a common point. Moreover, prove that the point of concurrency lies two-thirds along the length of each median.

The point of concurrency is called the *centroid* (or, to a physicist, the *centre of gravity*) of the triangle.

Answer: Call the vertices of the triangle A, B, and C and the midpoints of the sides M, N, and O as shown:



We have:

$$M = \frac{1}{2}B + \frac{1}{2}C$$
$$N = \frac{1}{2}A + \frac{1}{2}C$$
$$O = \frac{1}{2}A + \frac{1}{2}B$$

The point two-thirds away along the median connecting A to M is given by:

$$A + \frac{2}{3}\overrightarrow{AM} = A + \frac{2}{3}(M - A)$$
$$= A + \frac{2}{3}\left(\frac{1}{2}B + \frac{1}{2}C - A\right)$$
$$= \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$$

The point two-thirds away along the median connecting B to N is given by:

$$B + \frac{2}{3}\overrightarrow{BN} = B + \frac{2}{3}(N - B)$$
$$= B + \frac{2}{3}\left(\frac{1}{2}A + \frac{1}{2}C - B\right)$$
$$= \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$$

The point two-thirds away along the median connecting C to O is given by:

$$C + \frac{2}{3}\overrightarrow{CO} = C + \frac{2}{3}(O - C)$$
$$= C + \frac{2}{3}\left(\frac{1}{2}A + \frac{1}{2}B - C\right)$$
$$= \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$$

These are the same point! □

Exercise: Find the coordinates of the centroid of a triangle with vertices $A=(3,9)$, $B=(-2,2)$, and $C=(10,-8)$.

THE EQUATION OF A LINE

Exercise: Draw two points in the middle of a blank piece of paper and label them A and B. On your paper sketch the locations of the point following points:

a) $A+\overrightarrow{AB}$

b) $A+2\overrightarrow{AB}$

c) $A+3\overrightarrow{AB}$

d) $A+10\overrightarrow{AB}$

e) $A+\frac{1}{2}\overrightarrow{AB}$

f) $A-\overrightarrow{AB}$

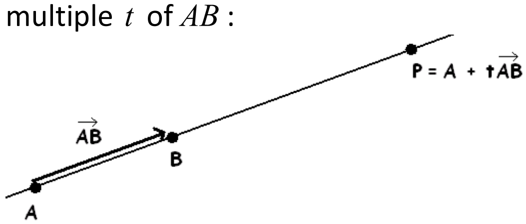
g) $A-2\overrightarrow{AB}$

h) $A-\frac{5}{2}\overrightarrow{AB}$

i) $A-100\overrightarrow{AB}$

This exercise shows the following:

Suppose A and B are two points in the plane. Then any point P on the line through A and B can be reached by starting at A and walking some multiple t of $\overrightarrow{AB}$:



Thus the "equation of the line through A and B" is:

$$A+t\overrightarrow{AB}$$

This is not the form of the equation one learns in a typical high-school algebra course: It is an equation that depends on a <u>parameter</u> t. If we like to think of t as time, then the expression $A+t\overrightarrow{AB}$ can be thought of as the location of a particle at time t. At $t=0$, the particle is at the point A, at time $t=1$ it is at $A+\overrightarrow{AB}=B$, and $t=2$ is to the right of B the same distance from B as A is, and at time $t=\frac{1}{2}$ the particle is halfway between A and B, that is, at their midpoint.

Example: Let $A=(1,6)$ and $B=(3,10)$. Then any point (x,y) on the line connecting A and B has coordinates given by:

$$(x,y)=A+t\overrightarrow{AB}=(1,6)+t<2,4>$$
$$=(1+2t,6+4t)$$

Thus we have:

$$x=1+2t$$
$$y=6+4t$$

It is fine to leave this as the equation of a line. However, to bring the equation into familiar form, solve for t in the first equation and substitute the result into the second. This yields:

$$y=6+4\left(\frac{x-1}{2}\right)=2x+4$$

Thus the equation of the line connecting A and B is $y=2x+4$.

Exercise: Let $A=(a_1,a_2)$ and $B=(b_1,b_2)$ and let $P=(x,y)$ be some point on the line through A and B. (Thus $P=(x,y)=A+t\overrightarrow{AB}$.)

a) Show that $t=\frac{x-a_1}{b_1-a_1}$ and also that $t=\frac{y-a_2}{b_2-a_2}$.

b) Show that we must have $y-a_2=\left(\frac{b_2-a_2}{b_1-a_1}\right)(x-a_1)$, which is the equation for the line as given in a typical algebra course.

c) The equations given in part *a*) of this exercise are invalid if either $b_1 - a_1 = 0$ or $b_2 - a_2 = 0$. What is the geometric interpretation of these statements? Is the equation $P = A + t\overrightarrow{AB}$ still valid in each of these situations?

d) Write an equation of the line $2y = 3x - 5$ in terms of some parameter t.

Exercise:

a) Show that the equation of a line through points A and B can also be written:

$$P = (1-t)A + tB$$

b) Put $t = 0$ and $t = 1$ into this formula and see which special points appear. Give a geometric interpretation of the set of all points $P = (1-t)A + tB$ for $0 \le t \le 1$.

Aside comment: Young students in a robotic class might very well be very familiar with parametric equations: "Where do you want the robot to be at time t? What x-coordinate and y-coordinate?" Why do we leave parametric equations to so late in the curriculum when they represent a very natural way to think of things?

I might even personally argue that the equation of a line as presented here is a natural way to think of these objects in general for beginning high-school students. What would an algebra curriculum based on this approach be like?

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