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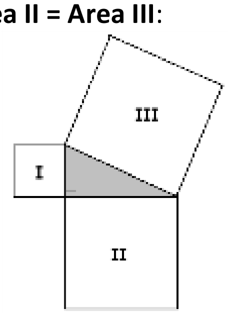
TANTON'S TAKE ON …

TRIG ADDITION FORMULAS

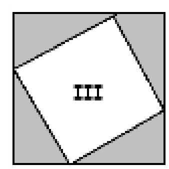
DECEMBER 2014

Here's a lovely, purely geometric, proof of the Pythagorean Theorem, believed to have originated in China possibly as early as 1100 B.C.E. This proof is today called "The Chinese Proof." It's fun as a physical demonstration:

We wish to show in the following picture that **Area I + Area II = Area III**:



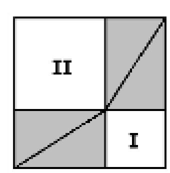
To do this, cut out four copies of the same right triangle and arrange them in a large square as shown.



Notice that the white space in the figure is precisely area III:

White Space = Area III.

Now arrange the triangles this way to see both areas I and II.



We observe:

White Space = Area I + Area II.

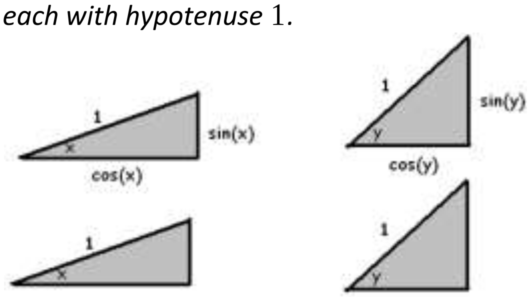
The area of the white space has not changed by rearranging the four triangles in the large square. We thus conclude:

Area I + Area II = Area III.

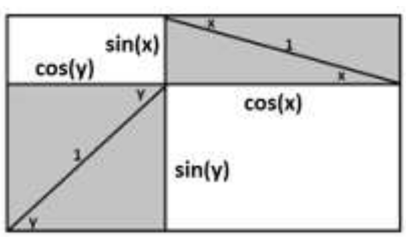
I love conducting this activity with students of all ages, showing them the first arrangement of four triangles in a square (cardboard cut-outs pinned to the wall) and challenging them to rearrange the triangles so that either area I or area II (or both!) appear. It takes an epiphany to discover the second arrangement.

Matters are particularly delightful if I later work with the same students in a pre-calculus trigonometry unit. We can discover trigonometric angle formulas by mimicking the very same proof!

Draw two copies each of two right triangles, each with hypotenuse 1.



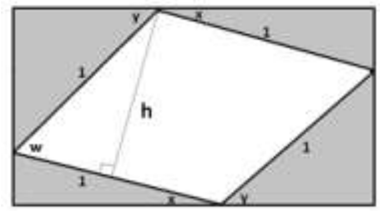
Arrange them into a rectangle as shown:



The area of the white space is the sum of the areas of two small rectangles:

White Space =
$$\sin(x)\cos(y)+\cos(x)\sin(y)$$

We can also rearrange these four triangles within the large rectangle as follows:



The white space is now a rhombus with side-length 1. *The area of a rhombus (in fact, of any parallelogram,) is "base times height." The base length is* 1 *and the height is the length* h *indicated. We see that* h *is the opposite edge of a right triangle of hypotenuse* 1 *and angle*
$$w=180-(90-x)-(90-y)$$
$$=x+y.$$

Thus:

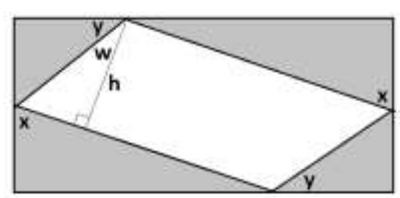
White Space =
$$1\times h=1\times\sin(x+y)=\sin(x+y).$$

It is the same white space. We have thus proved:

$$\boxed{\sin(x+y)=\sin x\cos y+\cos x\sin y}$$

(at least for positive acute angle x *and* y *.)*

In the same way, we can consider this variation of the proof. (Do you see the change on which angle is called x *?)*



It establishes:

$$\boxed{\cos(x-y)=\cos x\cos y+\sin x\sin y}$$

Lovely! The sense of continuity for students this approach supplies is terrific.

NATURAL NEXT QUESTIONS

I deeply worry about a curriculum that pushes students to results and not let mathematics be the organic conversation it deserves to be and which real authentic teaching demands! We have opportunities right now to let conversations evolve.

For example, students might now ask:

Is there a formula for $\sin(x-y)$ *?*

One for $\cos(x+y)$ *?*

A lovely problem-solving technique is to transform any given problem in hand to one you have solved before.

We have a formula for $\sin(x+y)$, the sine of a sum of two angles. Can we convert $\sin(x-y)$ to a sine of a sum? YES!

$$\begin{aligned}\sin(x-y) &= \sin(x+(-y))\\ &= \sin(x)\cos(-y)+\cos(x)\sin(-y)\\ &= \sin x\cos y-\cos x\sin y.\end{aligned}$$

Similarly,

$$\begin{aligned}\cos(x+y) &= \cos(x-(-y))\\ &= \cos(x)\cos(-y)+\sin(x)\sin(-y)\\ &= \cos x\cos y-\sin x\sin y.\end{aligned}$$

Fabulous!

SERIOUS PROBLEM: Unfortunately, this approach is not valid! We've only established the first two formulas for positive acute angles x and y. We are not permitted to apply these formulas to a negative angle, $-y$!

Bother!

But if we play with the formulas:

$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$

and

$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$

trying different values angles for x and y on a calculator, they seem to work nonetheless!

Moreover, they seem to hold for <u>all</u> types of angles, obtuse ones too! Have we obtained correct formulas by invalid means?

BIG QUESTION: Do the four formulas:

$$\begin{aligned}\sin(x+y)&=\sin x\cos y+\cos x\sin y\\ \cos(x-y)&=\cos x\cos y+\sin x\sin y\\ \sin(x-y)&=\sin x\cos y-\cos x\sin y\\ \cos(x+y)&=\cos x\cos y-\sin x\sin y\end{aligned}$$

actually hold for <u>all</u> angles x and y ?

Recall that we have only established the first two formulas for the case of x and y being measures of positive acute angles.

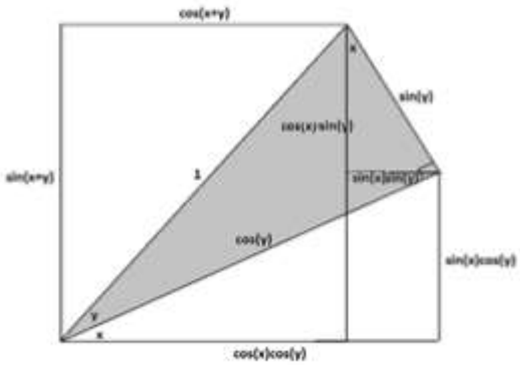
Now we have a super classroom mystery!

EXTENDING OUR RESULTS

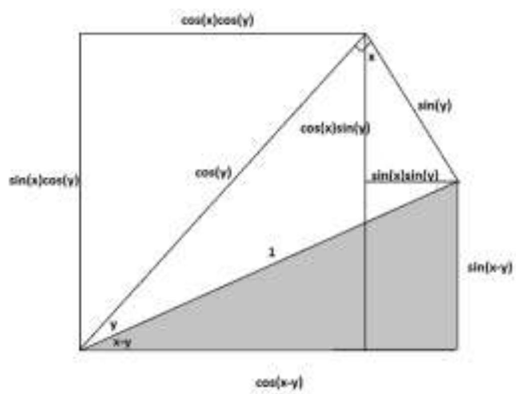
It seems our initial proof is simply not adequate. Is there a better way to get to results like these?

Why not, for homework, send students to the internet to hunt for proofs of these results? Find out what have other people done in the past. (Let's teach our students to search the literature!) Have students explain proofs they find and discuss any limitations these proofs might have.

For example, students might find these classic visual proofs of all four formulas for the case of acute angles:



$$\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

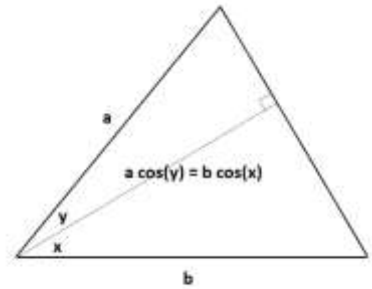


$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

Question: Can you see how these pictures prove the four formulas?

Extra: How could you modify the first diagram to allow for the possibility of $x+y$ having a measure greater than 90°?

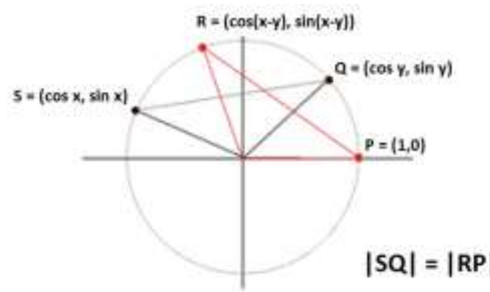
Or perhaps a gem like this:



$$Area = \frac{1}{2}ab\sin(x+y)$$
$$= \frac{1}{2}a(b\cos y)\sin x + \frac{1}{2}b(a\cos x)\sin y$$

Question: Can we extend this argument to obtuse triangles?

Or students might find standard textbook proof using the distance formula for rotated points on the unit circle:



Question: This picture establishes one of the four trigonometric addition formulas. Which one?

Can you see that this proof establishes that one formula as valid for <u>all</u> values x and y?

Can you then use this one formula, valid for all inputs, to establish the remaining three formulas? (The answer is yes!)

It can be a great class period discussing all the different proofs students find and working to push each proof to its limits.

FINISHINHG UP

It is not always obvious whether or not given proofs for the four formulas extend to non-acute angles x and y. But as soon as one has the formulas established for acute values, it is possible to extend the results to all other values as well. To do so, one must make use of the following transformation results for trigonometric functions:

$$\sin\left(x-90^\circ\right)=-\cos\left(x\right)$$

$$\sin\left(x+90^\circ\right)=\cos\left(x\right)$$

$$\cos\left(x-90^\circ\right)=\sin\left(x\right)$$

$$\cos\left(x+90^\circ\right)=-\sin\left(x\right)$$

along with symmetry results:

$$\sin\left(-x\right)=-\sin\left(x\right)$$

and

$$\cos\left(-x\right)=\cos\left(x\right).$$

For example, if we know the formula for $\sin\left(x+y\right)$ is valid for x and y each acute in measure, then we can ask: What if x is acute and y has measure between 90° and 180°? Then $y-90^\circ$ is an angle of acute measure and we are permitted to write:

$$\sin\left(x+y-90^\circ\right)$$
$$=\sin\left(x\right)\cos\left(y-90^\circ\right)+\cos\left(x\right)\sin\left(y-90^\circ\right).$$

This reads

$$-\cos\left(x+y\right)=\sin x\sin y-\cos x\cos y$$

which can be rewritten:

$$\cos\left(x+y\right)=\cos x\cos y-\sin x\sin y$$

thereby extending one of the four formulas to the case with one angle obtuse.

Continuing play this way we can extend the each of the four formulas to angles beyond just acute ones. (Covering all cases is mighty tedious though!)

PEDAGOGICAL COMMENT

The distance formula proof seems to be most popular among curriculum writers. It establishes:

$$\cos(x-y)$$
$$=\cos\left(x\right)\cos\left(y\right)+\sin\left(x\right)\sin\left(y\right)$$

as valid for <u>all</u> values x and y. Thus we are now permitted to replace y with $-y$ to obtain the formula for $\cos\left(x-y\right)$ valid for all x and y values. We are also permitted to replace y in each of these formulas with $y-90^\circ$ to obtain the formulas for $\sin\left(x+y\right)$ and $\sin\left(x-y\right)$.

It is true we can take students straight to this distance formula proof, but why the rush to get to these formulas and intellectual perfection?

Research in mathematics, or any real problem solving activity, rarely leaps to the "perfect solution" first. One makes partial steps, adequate in some circumstances and probably inadequate in others. The challenge then is to push ideas further, salvage ideas, transmute or even abandon approaches for new ones, and then later, usually much later, come up with a robust solution that does the complete job. Why don't we model that process for students too?

MATHEMATICAL ADDENDUM:

We have the angle addition formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Setting $y = x$ gives us the double angle formulas:

$$\begin{aligned} \sin(2x) &= 2\sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \end{aligned}.$$

The triple angles formulas follow too. For example:

$$\begin{aligned} \sin(3x) &= \sin(2x+x) \\ &= \sin(2x)\cos x + \cos(2x)\sin x \\ &= 2\sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x \\ &= 3\sin x \cos^2 x - \sin^3 x \end{aligned}$$

$$\begin{aligned} \cos(3x) &= \cos(2x+x) \\ &= \cos(2x)\cos x - \sin(2x)\sin x \\ &= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\ &= \cos^3 x - 3\sin^2 x \cos x \end{aligned}$$

With the use of the complex number i (which satisfies $i^2 = -1$) these two triple angle formulas can be united as a single identity:

$$\cos(3x) + i\sin(3x) = (\cos x + i\sin x)^3.$$

In fact, one can prove in general that for each positive integer n:

$$\boxed{\cos(nx) + i\sin(nx) = (\cos x + i\sin x)^n.}$$

Neat! (In both sense of the term!)

Comment: If one is aware of Euler's famous formula $e^{ix} = \cos x + i\sin x$, this is the statement:

$$e^{inx} = \left(e^{ix}\right)^n.$$

For a complete discussion of the Euler's formula and role of using complex numbers to simplify trigonometry see my text *THINKING MATHEMATICS! Vol 5: Slope, e, i, pi and all that.*

PEDAGOGICAL WARNING: Please don't reveal Euler's Formula in a pre-calculus class! (Why do we do this?) The discovery of this formula was a complete shocker in the story of calculus. It stunned Euler and his contemporaries! Let the discovery of Euler's Formula naturally unfold for your students when you teach calculus - and let it be a complete shocker for them too. Don't ruin a major mathematical surprise and incredible delight! (Why do we insist on stripping the human experience from the mathematics curriculum?)

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