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TANTON'S TAKE ON …

FACTORING

AUGUST 2017

I am personally very torn about the relevance of teaching factoring in the high-school curriculum – the factoring of quadratics, cubics, and maybe higher-order polynomials. The techniques we teach, such as "look for factors of ac that sum to b" and "chunking" are designed to work for examples for which those techniques happen to work. The truth is most polynomials don't factor nicely, if they factor at all - and I really do mean most!

Exercise: *There are 100 quadratics of the form* x^2+bx+c *if we allow* b *and* c *to each run through the single digits* 0 *through to* 9*. Show that only 21 of those quadratics factor over the integers.*

Research: *If we let* a *run through the integers 1 through* N *and each of* b *and* c *run through the digits from* $-N$ *to* N *, how many of the* $N(2N+1)^2$ *quadratics of the form* ax^2+bx+c *factor over the integers? Does the percentage of factorable quadratics tend to zero as* N *grows?*

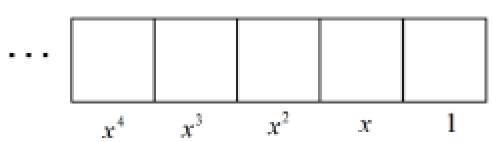
With the myriad of factoring examples we present to students, do we give the impression that most polynomials factor nicely?

One might argue that this point is moot: the point of factoring is not to factor, per se, but to develop general algebraic facility. Sure. But maybe there are alternative ways to encourage such fluency?

I'll put my pedagogical qualms aside, and bring forward a new idea. It is a possible visual approach to factoring that might or might not have some merit. My suspicion is that it too only works well for factoring problems we design for it to work. The same trap! But maybe, as an additional option for students to try, it can make the carefully constructed textbook problems straightforward – sometimes.

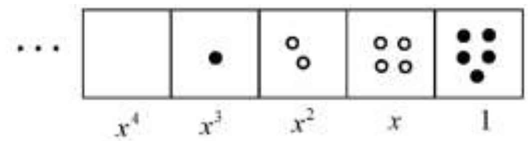
This work is based on the concept of Exploding Dots, my current obsession and rollout topic for Global Math Week 2017. (See www.theglobalmathproject.org/gmw and learn specifically about Exploding Dots at www.gdaymath.com/courses/exploding-dots/.)

Here's a picture of an $1 \leftarrow x$ machine.

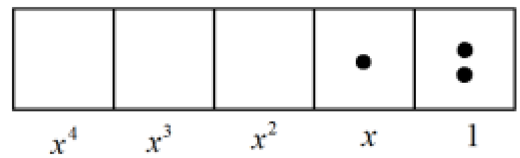


Each box represents a place-value in base x. (If x happens to have the value 10, then these are the usual units, tens, hundreds, … places for how we write numbers.) We use dots (solid dots) to represent positive units and antidots (open dots) to represent negative units.

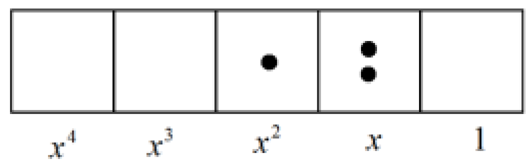
For example, here is the representation of $x^3 - 2x^2 - 4x + 5$ in the $1 \leftarrow x$ machine.



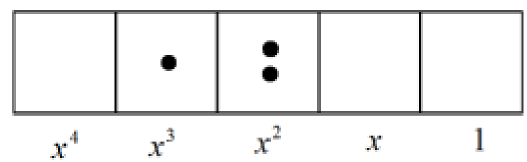
Here's what $x + 2$ looks like.



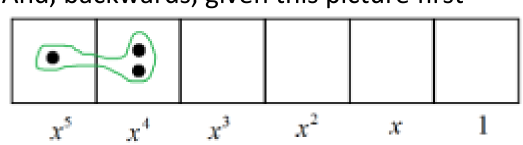
And here's $x(x+2) = x^2 + 2x$,



and $x^2(x+2) = x^3 + 2x^2$.



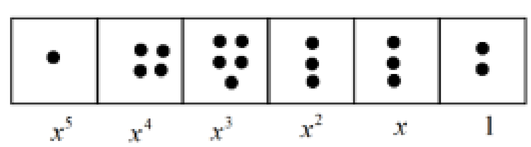
And, backwards, given this picture first



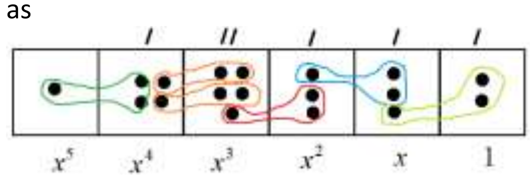
we can see that it is $x^4(x+2)$.

And now we can see this picture of

$$x^5 + 4x^4 + 5x^3 + 3x^2 + 3x + 2$$



as



We read this picture as

$$x^4(x+2)+2x^3(x+2)+x^2(x+2)$$
$$+x(x+2)+(x+2)$$

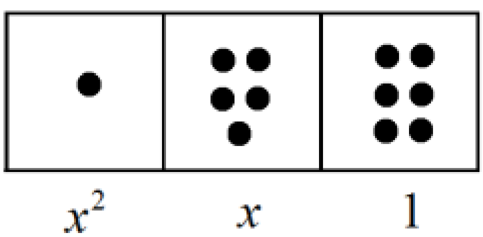
and so we have factored out a linear term of our polynomial.

$$x^5+4x^4+5x^3+3x^2+3x+2=$$
$$(x+2)(x^4+2x^3+x^2+x+1).$$

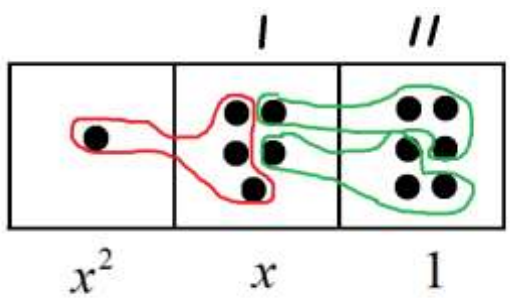
Upshot: *If we draw a picture of a polynomial in an* $1 \leftarrow x$ *array and recognize repeated components within the picture, then we have factored the polynomial!*

Sometimes repeated components can be spotted.

Example: Factor x^2+5x+6.

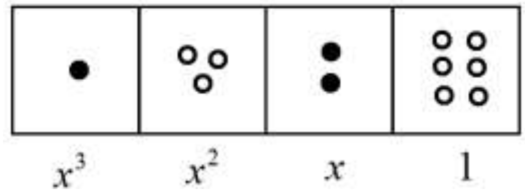


Answer: I see

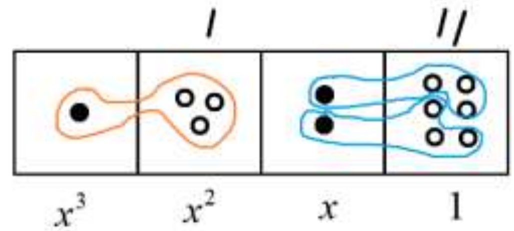


This is $x(x+3)+2(x+3)$, which is $(x+2)(x+3)$. (Could you start by seeing something different?)

Example: Find a linear factor of x^3-3x^2+2x-6.



Answer: I see



This is $x^2(x-3)+2(x-3)$, which is $(x-3)(x^2+2)$.

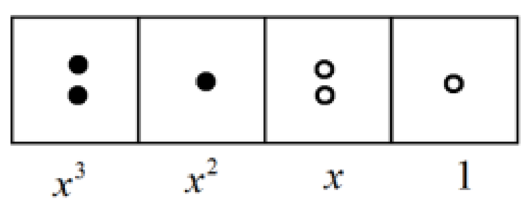
Comment: This feels a bit clunky. If you first discuss the division of polynomials as done in lessons 6.2 and 6.4 of www.gdaymath.com/courses/exploding-dots/, then it is very natural to read this picture instead as

$$\frac{x^3-3x^2+2x-6}{x-3}=x^2+2$$

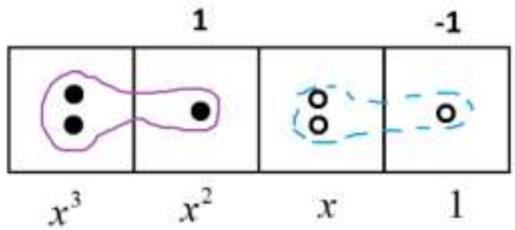
from which it follows that

$$x^3-3x^2+2x-6=(x-3)(x^2+2).$$

Example: Find a linear factor of $2x^3+x^2-2x+1$.

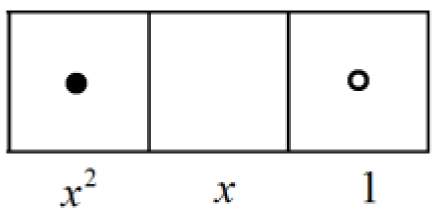


Answer: I see a copy of $2x+1$ and an anti-copy of $2x+1$!

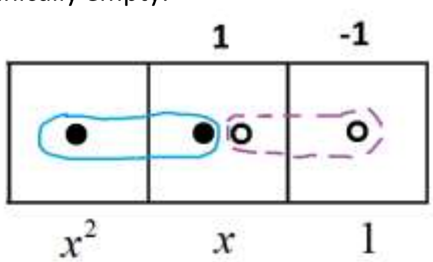


We have $\frac{2x^3+x^2-2x-1}{2x+1}=x^2-1$ and so $2x^3+x^2-2x-1=(2x+1)(x^2-1)$.

Example: Go all the way! Factor x^2-1.

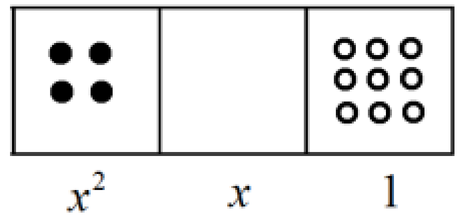


Answer: This is looking troublesome! But if you have worked through the problem with division in lesson 6.4 of Exploding Dots you will likely think to add a dot/antidot pair to the middle box while keeping that box technically empty.



We see $\frac{x^2-1}{x+1}=x-1$ and so $x^2-1=(x-1)(x+1)$.

Challenge: Can you see how to factor the difference of two squares $4x^2-9$?



How many dot/antidot pairs should we add to the middle box?

HONESTY STATEMENT: It is not clear to me if this visual approach is particularly helpful for factoring quadratics, in general. One can sometimes luck out and see repeated blocks (draw a picture of $4x^2-8x+3$), but often factoring a quadratic requires adding some clever number of dot/antidot pairs. (Try factoring $8x^2-2x-3$.)

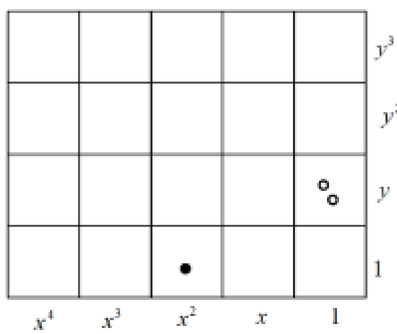
However, it seems to me that this visual approach is fabulous for most textbook questions about finding linear factors of cubic equations. They tend to be visually obvious.

Non-Obvious Cubic Challenge: Can you spot a linear factor of the first polynomial we presented, x^3-2x^2-4x+5?

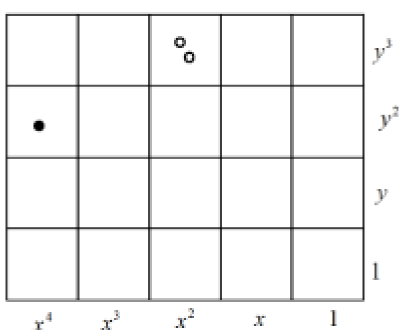
GOING FURTHER

Global Math Project Ambassador Kiran Bacche had the brilliant insight to take this work to two dimensions! I'll give you just a teaser on his work here. [Kiran is writing up all his thoughts and insights in a document for the Global Math Project. Check the website to see it there soon!]

Here's a picture of $x^2 - 2y$.

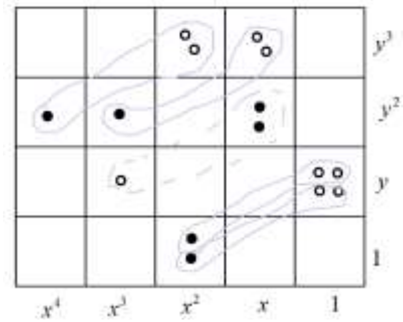


And here is a picture of $x^2y^2\left(x^2-2y\right)$. (Note where the dot and pair of antidots are in relation to the x^2y^2 cell.)



Here's is a picture of

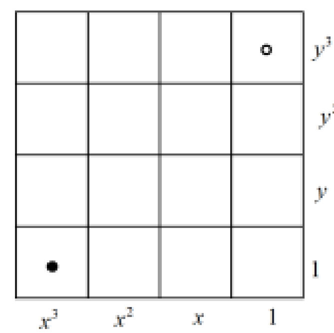
$$x^4y^2 + x^3y^2 - x^3y - 2x^2y^3 + 2x^2 - 2xy^3 + 2xy^2 - 4y$$



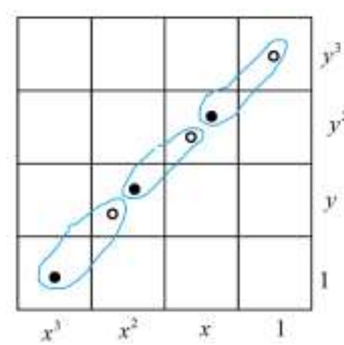
which we can now just see as

$$\left(x^2-2y\right)\left(x^2y^2+xy^2-xy+2\right).$$

Or consider the difference of two cubes, $x^3 - y^3$.



It seems irresistible to add dot/antidot pairs.



We see that

$$x^3 - y^3 = \left(x-y\right)\left(x^2+xy+y^2\right).$$

Now factoring seems interesting and fun to me. Thank you Kiran!

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