TANTON'S TAKE ON ...

EXPLAINING FACTORING

AUGUST 2016

Let's continue the theme of last month's essay, http://www.jamestanton.com/wp-content/uploads/2012/03/Curriculum-Essay_July-2016_Factoring-Trinomials1.pdf and explain the classic "split the middle term" technique for factoring quadratics. We'll see that it relies on a famous – and deep - piece of mathematics that is typically not explained in schools.

Recall, to factor ax^2+bx+c, with a, b, c integers, the technique says to look for a pair of factors p, q of the product of the first and last coefficients, a and c, that sum to the middle coefficient b.

$$p+q=b$$
$$pq=ac$$

If you can find such a pair of integers, then you are good to go! Just rewrite the quadratic as $ax^2+px+qx+c$ (the middle term is split), pull out a common factor from first and second terms and another from the third and fourth terms, and a factorization will magically fall right out.

For example, to factor $6x^2-13x-5$, look for a pair of factors of $6\times(-5)=-30$ that sum to -13. I think of -15 and 2. Then

$$\begin{aligned}6x^2-13x-5&=\underline{6x^2-15x}+\underline{2x-5}\\&=3x\cdot(2x-5)+1\cdot(2x-5)\\&=(3x+1)(2x-5).\end{aligned}$$

Indeed, magic!

Question: *Does changing the order of the factors affect matters? Is working with $ax^2+qx+px+c$ sure to give the same* *factorization as working with* $ax^2 + px + qx + c$*?*

Question: *If one cannot find a pair of integers* p *and* q *that work the desired way does mean that the quadratic cannot be factored over the integers?*

WHOA! TOO MUCH WRITING!

Once you've found your p and q don't bother splitting the middle term. Just write down

$$\frac{(ax+p)(ax+q)}{a}$$

and you are done!

For $6x^2 - 13x - 5$ we have $p = -15$, $q = 2$ and we get

$$\frac{(6x-15)(6x+2)}{6} = \frac{(6x-15)}{3} \cdot \frac{(6x+2)}{2}$$
$$= (2x-5)(3x+1).$$

For $10x^2 + 83x + 60$ think $p = 75$, $q = 8$ and write

$$\frac{(10x+75)(10x+8)}{10}$$
$$= (2x+15)(5x+4).$$

Isn't magic wonderful?

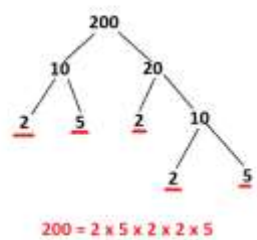
Question: *Try this method for* $x^2 - a^2$*, a difference of two squares. What do you get? (What does this method give you for* $x^2 - 2$*? For* $x^2 + 2$*?)*

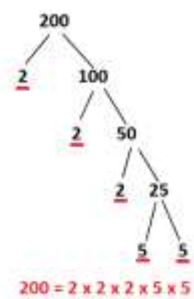
BURNING QUESTION: *Why do these methods work?*

FACTORING INTEGERS

Every number factors in a trivial way as a product of 1 and itself. A positive integer that factors in a non-trivial way is called *composite*.

In our early grades we were taught to draw factor trees of numbers to break a given number down into a product of *primes*: positive integers greater than 1 that are not composite.





(It is handy here to <u>not</u> regard 1 as a prime number - otherwise one would never know when to stop factoring in a factor tree.)

It is truly remarkable that no matter what choices one makes along the way in drawing a factor tree for a given number, in the end, the same list of primes is sure to appear at the base of the tree.

Question: *You and a friend each decide to draw factor trees for the number* 18006402000*. Are you certain that you will each see the same list of primes at the bases of your trees (though the order of those primes might be different)? How do you know?*

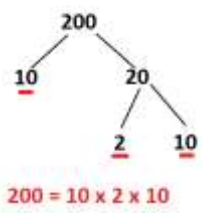
WELCOME TO EVENESTAN

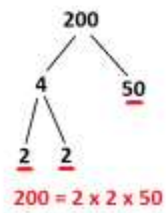
This previous question is a very serious one, as the folk of Evenestan will attest.

In this land, people are aware only of the even numbers. Ask an Evenestanian to count to ten and she will say: 2,4,6,8,10. If you ask her about 1, 3, 5, 7, and 9, she will only stare at you blankly.

In Evenestan, some numbers break down into factors. For example, $12 = 2 \times 6$ and $16 = 4 \times 4$. Such numbers are called *e-composite.* Other numbers like 6 and 18 do not factor in Evenestan. (Remember, odd integers do not exist here.) Such numbers are called *e-prime.*

As in the U.S., young children in Evenestan are taught to draw factor trees.





But these children are fully aware that different choices made in constructing factor trees can lead to different e-prime factorizations. Factor trees are not unique!

Challenge: *Actually, factor trees in Evenestan are unique up to a certain point. For example, the factor trees of the numbers 2, 4, 6, 8, …, 20 are fixed. What is the first number in Evenestan that yields two different e-prime factorizations?*

Children from Evenestan are very confused when they move to the U.S. and learn about factor trees with even and odd numbers. Why does everyone here assume it "obvious" that the base numbers of factor trees are essentially unique? It is not at all true in Evenestan. Maybe all the factor trees with even and odd numbers are unique up to a certain point – for all the small examples one does in class, for instance – and then start varying for bigger numbers? Maybe the number 18006402000 is large enough to now give two different prime factorizations?

THE FUNDAMENTAL THEOREM OF ARITHMETIC

It turns out that our system of arithmetic, with both even and odd counting numbers, is remarkably special.

Over 2000 years ago, the great Greek scholar Euclid discussed the notions of prime and composite numbers, proved that there are an infinite number of prime numbers, and established that they do indeed serve the role of being the atoms of arithmetic. Euclid proved that each and every counting number breaks down into a product of one, and only one, finite list of primes (though one may vary the order in which you write that list). That is, Euclid proved what everyone just assumes as true in, in fact true: the base numbers of a factor tree for any given number are unique.

This result is so fundamental and so important in mathematics that it is called the *Fundamental Theorem of Arithmetic*.

But it takes some doing to set up a proof of this result. I present one here, http://www.jamestanton.com/?p=695, though you might want to check out my video on the Euclidean Algorithm and Jug Filling first http://www.jamestanton.com/?p=692.

The Fundamental Theorem of Arithmetic tells us, for example, that

* $11 \times 11 \times 17 \times 37 \times 43$ cannot equal $13 \times 13 \times 17 \times 97$. (If they were equal, then we'd have two different prime factorizations of the same number)

* No non-trivial power of 7 can equal a power of 17 (if $N = 7^a = 17^b$, then we'd have two different prime factorizations of the same number)

* If nm is a multiple of 47 then at least one of n or m is itself a multiple of 47.

(The prime factorizations of n and m combine to give a prime factorization of nm. And if $nm = 47k$, then the prime factorization of k gives another one. The prime 47 must therefore must appear in at least one of the two prime factorizations of n and m.)

Challenge: *Prove that no non-trivial power of 12 can equal a power of 14. Prove that no non-trivial power of 12 can equal a power of 8. Can a non-trivial power of 12 equal a power of 6?*

Challenge: *Prove that there are no integers a and b that make the equation $a^2 = 2b^2$ true. (Hence deduce that $\sqrt{2}$ is an irrational number.) Prove that $\sqrt[5]{8}$ is irrational. Develop a general theory as to when $\sqrt[n]{M}$ is rational.*

FACTORING QUADRATICS

We are now ready to prove

Theorem: $ax^2 + bx + c$, *with a, b, c integers, factors as a product of two linear terms with integer coefficients if, and only if, there are integers p and q such that*

$$p + q = b$$
$$pq = ac$$

And, if this is indeed the case, the two factoring techniques presented in this essay are guaranteed to work.

Proof: Here goes!
One direction of thinking is straightforward.

Suppose the quadratic factors over the integers. That is,

$$ax^2 + bx + c = (rx + s)(ux + v)$$

with r, s, u, v integers. Then

$$ax^2 + bx + c = rux^2 + (rv + su)x + sv.$$

Set $p = rv$ and $q = su$, then we have a pair of integers with

$$p + q = rv + su = b$$
$$pq = rvsu = ac,$$

as desired.

The reverse direction of thinking is the tricky part.

Suppose we have found a pair of integers with p and q with

$$p + q = b$$
$$pq = ac.$$

We now want to deduce that $ax^2 + bx + c$ factors.

Look at

$$ax^2 + px + qx + c.$$

Suppose d is the largest common factor of a and p, and e is the largest common factor of q and c. "Pull out" these common factors and write

$$a = dA \qquad q = eQ$$
$$p = dP \qquad c = eC$$

for some integers A, P, Q, C.

Now, A and P have no primes in common (as we pulled out the largest common factor from a and p). Nor do Q and C. But from the equation $pq = ac$ we get

$$dePQ = deAC$$

that is,

$$PQ = AC$$

with P and A sharing no common primes, and Q and C sharing no common primes. So, all the primes in the prime factorization of A must be sitting as part of Q, and all the primes in the prime factorization of Q must be sitting as part of A. That is, A and Q must have the same prime factorization, and so $A = Q$. And it then follows that $P = C$, though one can apply the same prime factorization argument here too.

(Notice we are relying here on the uniqueness of the prime factorization of a number: the primes appearing in the products PQ and AC must match. For students of Evenestan, this cannot be guaranteed and so all bets are off on telling when a quadratic factors in Evenestan.)

All right. Where are we?

We have $A = Q$ and $P = C$, so

$$a = dA \qquad q = eA$$
$$p = dC \qquad c = eC$$

with

$$p + q = b$$
$$pq = ac.$$

Then

$$\begin{aligned} ax^2 + bx + c &= ax^2 + px + qx + c \\ &= dx(Ax + C) + e(Ax + C) \\ &= (dx + e)(Ax + C) \end{aligned}$$

So the quadratic factors over the integers, and the "split the middle term" technique works!

Going further we see

$$\begin{aligned} ax^2 + bx + c &= (dx + e)(Ax + C) \\ &= \frac{(dAx + eA)(dAx + dC)}{A \cdot d} \\ &= \frac{(ax + q)(ax + p)}{a} \end{aligned}$$

showing that the second technique works too.

Phew!

Comment: Now we can see why these techniques work is never explained in schools! (So is offering it as just "magic" the right thing to do?)

SOMETHING EXTRA

Consider again a quadratic $ax^2 + bx + c$ with integer coefficients.

The quadratic formula shows that if its discriminant $b^2 - 4ac$ is a perfect square, then the roots of the quadratic are sure to be rational numbers, but not necessarily integers. But does this nonetheless mean that the quadratic factors over the integers?

Certainly any quadratic that factors over the integers has rational roots: If $ax^2 + bx + c = (rx + s)(ux + v)$, then its roots are $-\frac{s}{r}$ and $-\frac{v}{u}$. So how about the converse?

Certainly any quadratic with rational roots $-\frac{s}{r}$ and $-\frac{v}{u}$ must be of the form $d\left(x + \frac{s}{r}\right)\left(x + \frac{v}{u}\right)$ for some number d. If this is meant to match a quadratic with integer coefficients, does this mean we could have written the factorization as $d'(rx + s)(ux + v)$ for some other number d', that is, as a factorization over the integers?

The answer is yes!

Theorem: *A quadratic $ax^2 + bx + c$ with integer coefficients factors over the integers if, and only if, $b^2 - 4ac$ is a perfect square.*

Consequently, a quadratic with integer coefficients factors over the integers if, and only if, it has rational roots.

Proof: Suppose $ax^2 + bx + c$ factors as

$$ax^2 + bx + c = (rx + s)(ux + v).$$

Then $a = ru$, $b = rv + su$, $c = sv$ and

$$b^2 - 4ac = (rv + su)^2 - 4rsuv$$
$$= (rv - su)^2$$
is a perfect square.

Suppose first, one the other hand, that $b^2 - 4ac$ is a perfect square.

We're hoping to find a pair of integers p and q that satisfy

$$p + q = b$$
$$pq = ac$$

as this will then show, as per the work of this essay, that the quadratic factors.

Can we get a feel for what p and q should be?

Well, we have here a system of two equations in two unknowns. Solving gives

$$p = \frac{b - \sqrt{b^2 - 4ac}}{2}$$

and

$$q = \frac{b + \sqrt{b^2 - 4ac}}{2} \quad ,$$

and one can check that these two values do sum to b and have product ac. (It is curious how reminiscent of the quadratic formula these each are!) However, we are not sure that p and q are integers.

But we are assuming that $b^2 - 4ac$ is a perfect square. And that turns out to be enough!

Suppose

$$b^2 - 4ac = n^2$$

for some integer n.

If n is even, then $b^2 - 4ac$ is even, meaning b^2, and hence b, is also even. If n is odd, then $b^2 - 4ac$ is odd, meaning b^2, and hence b, is also odd. So b and $n = \sqrt{b^2 - 4ac}$ are either both even or both odd.

Thus

$$p = \frac{b - \sqrt{b^2 - 4ac}}{2}$$

and

$$q = \frac{b + \sqrt{b^2 - 4ac}}{2}$$

are indeed integers, and ones with the required properties to allow the quadratic to factor. We have it!

Example: The quadratic $6x^2 + 5x + 1$ has $b^2 - 4ac = 25 - 16 = 9$ a perfect square and so it has rational roots (it does, $x = -\frac{1}{2}$ and $x = -\frac{1}{3}$) and it factors over the integers (and it does: $(2x+1)(3x+1)$).

Comment: The great mathematicians Carl Friedrich Gauss had much to say about polynomials with integer coefficients that do or don't factor over the integers. Look for *Gauss' Lemma* on the internet.

IN SUMMARY

The discriminant $b^2 - 4ac$ of a quadratic not only tells us the number of roots of the quadratic (two, one, or zero depending on whether the discriminant is positive, zero, or negative, respectively), but, for an integer quadratic, it also tells us about factorability.

An integer quadratic factors over the integers only if $b^2 - 4ac$ is a perfect square.

Lovely!

Is this observation well known?

SOME COMPUTER FUN

Here's something interesting:

Let a, b, c each run through the numbers 1 to 9. Of the 729 triples (a,b,c), it turns out that only 57 of them have b^2-4ac a perfect square. This means only $\frac{57}{729}\approx 7.8\%$ of the quadratics ax^2+bx+c with a,b,c positive single digits factor over the integers.

Very few!

This doesn't match the impression textbooks give about how often quadratics happen to factor over the integers.

Computer Challenge:
Let a, b, c each run through the integers $-N$ to N, but avoid $a=0$. Let $P(N)$ be the percentage of the $2N(2N+1)^2$ triples that have b^2-4ac a perfect square.
Does the value of $P(N)$ seem to go to zero as N grows? [Theory Challenge: Can you prove this is so?]

Another Computer Challenge:
Let's set $a=1$ so that we're working only with quadratics of the form x^2+bx+c. Can we plot a graph of all points (b,c) that yield a quadratic that factors?

We saw that x^2+bx+c factors if, and only if, we can find an integer p such that

$$p(b-p)=c.$$

This means for each integer b there are infinitely many values of c that yield a factorable quadratic. They come from letting p march through the values $0,1,2,3,\ldots$ and through $-1,-2,-3,-4,\ldots$

For example, for $b=10$, we get

$$c=5\cdot 5,\ 4\cdot 6,\ 3\cdot 7,\ 2\cdot 8,\ 1\cdot 9,\ 0\cdot 10,\ (-1)\cdot 11,\ (-2)\cdot 12,\ \cdots$$

and so we can plot the points

$(10,25),(10,24)$, $(10,21)$, ...

[Can you see why, in general, the highest point in this list is $\left(b,\frac{b^2}{4}\right)$ if b is even, and it is $\left(b,\frac{b^2-1}{4}\right)$ if b is odd?]

If we do this for a range of values of b what pictures result?

Alternatively: For each value of p the equation $p(b-p)=c$ is the equation of a line of slope p through the point $(0,-p^2)$ in the bc-plane. Is it easier to plot the integer points on each of these lines?

Question: For each value of c, p must be a positive or negative factor of c. This means that the number of points on each horizontal slice of the graph we create is related to the number of factors of c. (Does it match precisely the number of factors of c?)

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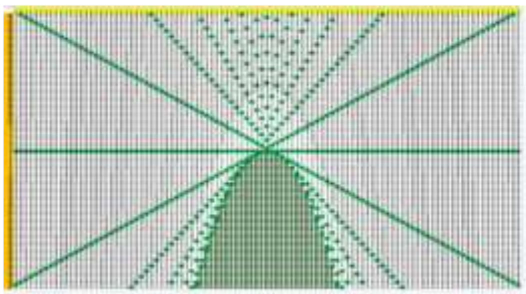
ADDENDUM

Twitterers @MathsPadJames, @mathforge, and @MrMattock informed me that they played with this work in a May 9th #mathschat and #mtbos discussion. Using Excel, they produced the following graph.



Here b and c each range from -50 to 50 with the horizontal axis the b axis and the vertical axis the c axis. But they have the positive values of c heading downwards. (So their graph is upside down to the set-up I laid out in the previous box.) Note too that the scaling is not square.

Also, it looks like there is a shaded U-shaped region in the positive c region. This is the region for which $b^2 - 4c$ is negative, that is, where the quadratic $x^2 + bx + c$ has no real roots and only factors over the complex numbers. (This region is bounded by the parabola $c = \frac{1}{4}b^2$.)

Can you see the lines of different slopes?

Can you see vertical cross-sections with a highest (here, lowest) point?

Can you see the horizontal cross-sections possibly matching the number of factors of c?