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# *TANTON'S TAKE ON …*

# SLOPE

**APRIL 2015**

Life is full of challenges and vague questions with no clear answers (and certainly no printed clarifications and solutions at the back of a textbook). All we can do in life is flail, hopefully with intelligence, common sense, and grace. And when we are truly stymied, all we can do is do something nonetheless. (This is surprisingly hard!)

These are the only two over-arching problem-solving strategies in life, and in mathematics: engage in successful flailing and to just do something! We must teach these techniques. It is incumbent upon us as teachers of mathematics to do so!

So this means we shouldn't always be experts for our students. We can't always come into class with previously constructed, well-defined questions and programmatic answers in hand. Instead, we must orchestrate opportunities for "flailure:" opportunities to forge paths through ambiguity, half-thinking, and haziness. We must help students develop the confidence to just do something when stymied, to try out ideas - any ideas - and see what does or doesn't come of them. We must model what it means not to know and to do something about it, to recognize hazy half-knowing, to not be satisfied with it, and to work to clear away the fog.

There are plenty of opportunities for this in the classroom.

I was recently asked to share my thoughts on teaching the concept of slope to middle-schoolers. I, of course, recalled what I did in my classes. Discussions and activities typically went something like this.

*Okay ladies and gentlemen, I have a question: How steep are the stairs at the end of the corridor just outside our classroom? Here are some rulers. Can you give me an answer?*
And we head to the stairwell.

So there's my vague question (and it is accompanied with absolutely no instruction as to what one might do with rulers). I've always felt that this question is appropriate as we each do possess some kind of internal sense of "steepness." We can all say, for example, that one set of stairs is steeper than another without any quantifiable measure of steepness in mind. Sometimes I felt it would be helpful to point this out to a group:

*Are the stairs inside the auditorium as steep as these ones?* Everyone agrees the other set of stairs I am referring to are significantly less steep. *Alright then. So what is "steepness"? How do we measure it? How steep are these stairs right here?*

Usually there is no action for a while as students stand there somewhat stymied. But invariably some brave soul decides to just do something: he or she measures the width of each step and each rise between steps.

At this point comes lots of praise from me for just doing something! We don't yet know whether or not this action will be helpful, but at least something is now in hand and that's always a good start to making sense of a question.

For the sake of this essay, let's say the width of each step is $8$ inches and the rise of each step is $6$ inches.

*Okay. Let's think about the information we have so far and see it is helpful.*

*You are telling me that if we move $8$ inches forward on these stairs, we move $6$ inches up.*

*So… if we move $16$ inches forward, we rise a total of … ?*
*Answer: $12$ inches.*

*And if we move $80$ inches horizontally forward, we rise a total of …?*
*Answer: $60$ inches.*

*Do these numbers tell me anything about "steepness"?*

Typically, for my classes, a student chimes in with a comment of the type:
*That's is an overall rise of $\frac{6}{8}=0.75$ inches for every one inch moved forward horizontally.*

They've been taught about unit rates of change. But I am not fully convinced this is meaningful to all students, so we head over to the "shallow" stairs in the auditorium. Students measure steps $20$ inches wide with a rise of $5$ inches between them, say.

At this point a conversation naturally ensues, one that compares an average rise of $0.75$ inches per horizontal inch forward for the first staircase with an average rise of $0.25$ inches per horizontal inch forward for the second set of stairs. That is, the notion that "rise over run" serves as a measure of steepness naturally comes into form.

So with this sense of success and satisfaction in hand, I then, of course throw a spanner in the works. (What's the U.S. version of this phrase? Something about wrenches?)

*Ummm …. We measured steepness from the bottom of this staircase going up and got the answer $0.25$ inches rise per horizontal inch of motion (on average).*

*Suppose Lulu wants to start at the top of the stairs and move downwards. From Lulu's perspective, what do you think the steepness of the stairs should be?*

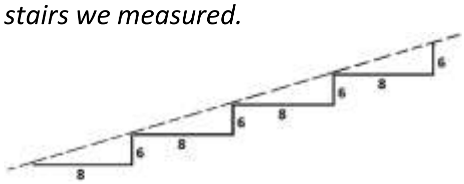
Some students answer that the steepness shouldn't change – they are the same set of stairs after all. Some students argue that Lulu's "rises" are actually negative, that she rises, on average, $-0.25$ inches for every inch moved forward.

All is now set to collectively decide that it is probably best to indeed consider decreases in height as "negative rise," and so "steepness" can be both positive and negative, depending on which perspective you choose to follow: are you heading up or heading down the stairs?

We're in good intellectual stead now.

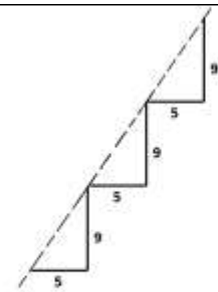
If you like, some problems of the following type can be fun.

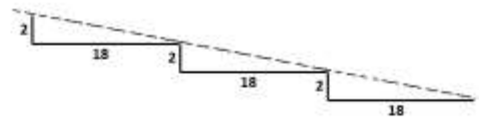
**1**. *Here is a picture of the first set of stairs we measured.*




*Assuming we are at the bottom of the stairs moving to the right, up the stairs, we gave the measure* $\frac{6}{8}=0.75$ *for their steepness (that is, a* $0.75$ *inch rise per each horizontal inch forward).*

*Following this idea, what measure of steepness might we assign to each of the following sets of stairs? (Assume we move from left to right in each picture.)*




**2.** *Draw and label a picture of a staircase of steepness 3. Draw one of steepness* $-3$. *Now draw one of steepness* $1/3$, *and finally one of steepness* $-1/3$.

**3.** *Draw and label a picture of a staircase of steepness* $1$.
(This is surprising to students – especially if you give steepness as a percentage measure!)

**4.** *Draw and label as best you can a picture of a staircase of steepness zero.*
(It takes some thinking!)

**5.** *Draw and label as best you can a picture of a staircase of steepness* $10,000$.

**6.** *Draw and label as best you can a picture of a staircase of steepness* $-10,000$.

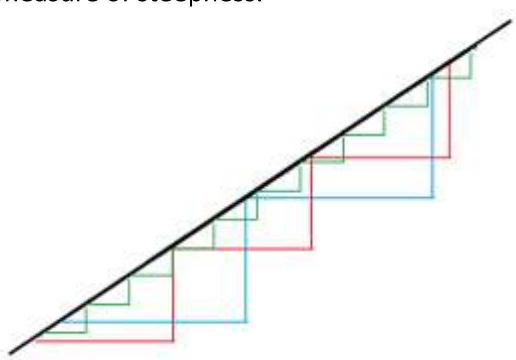
**7.** *Is it meaningful talk of staircase of infinite positive steepness? Of infinite negative steepness?*

**LINES AND SLOPE:**

One trouble with geometry courses is that they rarely define by what it means for a line to be straight. We each have an image in our mind as to what a straight line is, even though not one of us has ever actually seen one in the real world! (You might say that the edge of the table is straight, but it clearly won't be under an electron microscope. You might say that if you walked directly east you'd be walking in a perfectly straight line – but what will you say when you return to your starting point eight months later?)

There is something in our brains that we all like to believe is true about straight lines. What? (And it is surprising to me that we all seem to have the same notion programmed into our brains!)

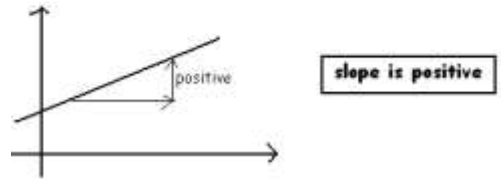
We all feel that straight lines have the same "steepness" no matter where on the line you choose to measure it. In terms of staircases, this means that all staircases draw under a given line will give the same measure of steepness.

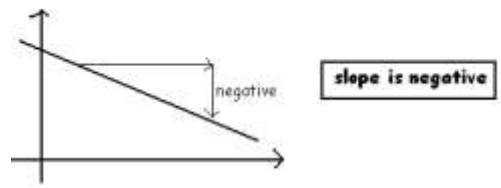


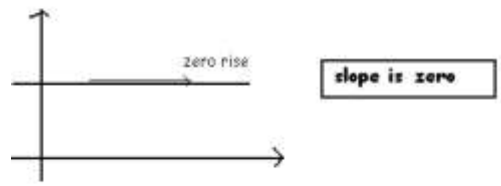
We are now set to discuss slope as the mathematical word for "steepness" and that slope is thus, naturally, given as "rise over run" for any right triangle you care to draw under the line.

**Comment:** We must point out the social convention that, in graphing, the positive horizontal axis is always placed to the right, and when examining slopes of lines, we follow the convention of assuming we are moving across the picture from left to right.

With this convention we have:

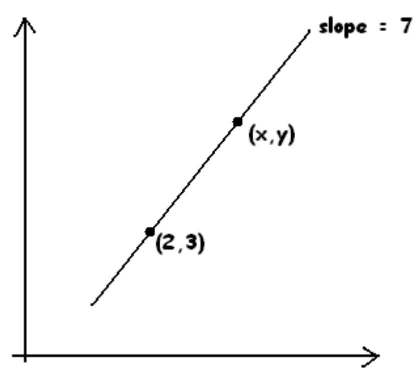



**"THE" EQUATION OF A LINE:**

*Suppose I tell you that a line passing through the point* $(2,3)$ *has slope* $7$ *. What can we say about the coordinates of any other point* $(x, y)$ *on that line?*




We like to believe that the slope of a straight line, no matter how we care to compute it, should have the same constant value. In this example that constant value is $7$ . So let's compute the slope using "rise over run" for the only two points we're thinking about: the specific point $(2,3)$ and the general point $(x,y)$. This gives the equation:

$$\frac{y-3}{x-2}=7.$$

DONE! Here is an equation that must be true for two values $x$ and $y$ to give a point $(x,y)$ that lies on the line.

And this equation is lovely. We can answer all sorts of questions about the line using it.

*Is the point $(10,19)$ on the line?*

Well, does $x=10$ and $y=19$ fit the equation?

$\frac{19-3}{10-2}=\frac{16}{8}=2$ . This is not $7$.

No! $(10,19)$ is not on the line.

*Is the point $(4,17)$ on the line?*

Yes! It fits the equation.

*What is the $y$-intercept of the line?*

Let's put in $x=0$. The equation tells me that the matching $y$-value satisfies: $\frac{y-3}{-2}=7$.

This gives the $y$-intercept $y=-11$.

*What is the $x$-intercept of the line?*

Putting $y=0$ gives $\frac{-3}{x-2}=7$ yielding $x=\frac{11}{7}$.

**Comment:** By <u>an equation of a line</u> we mean ANY equation in $x$ and $y$ that must be true for the point $(x,y)$ to be on the line. We found an equation, textbooks don't give this form a name, and I am not fussed about that one whit!

**Comment:** … But there is a caveat to my equation. Clearly $(2,3)$ is a point on the line, but the equation $\frac{y-3}{x-2}=7$ breaks down for $x=2$ and $y=3$.

However, if I apply a tiny piece of algebra to my equation and rewrite it as:

$$y-3=7(x-2)$$

now even $x=2$, $y=3$ gives a true number statement, and all is good.

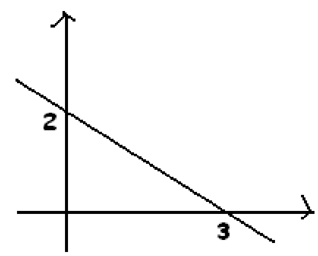
I will, for this reason, suggest we clear the denominators of the expressions we naturally first write down for an equation of a line. That is, I will first write:

$$\frac{y-p}{x-q}=m$$

but will later rewrite this as

$$y-p=m(x-q).$$

**PRACTICE:** *Write down an equation in $x$ and $y$ that must be true for a point $(x,y)$ to lie on this line:*




**Answer:** We have a line of slope $-2/3$ passing through the point $(3,0)$. The equation $\frac{y}{x-3}=-\frac{2}{3}$ will do. (Well, make that $y=-\frac{2}{3}(x-2)$.)

**Question:** Do we obtain a different equation if we focus on the point $(0,2)$ instead?

**LATER: BACK TO THE STAIRWELL**

*Give me a measure, an actual numerical value, of the steepness of these stairs. But this time you only have this straight length of wood (no markings) and a protractor to measure that number.*

Let's now discover and explore relationships between "rise over run," angles of elevation, and the tangent function!

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