WHOPPING COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

APRIL 2013

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

***

Looking for "Serious Mathematics Infused with Levity"? Check out http://www.mathily.org, a great summer-school program for math-enthusiastic students. The time to apply for it is now! Share this link with all your high-school scholars.

PUZZLER: (*A mathy puzzle today!*)

A polynomial is a function of the form:

$p(x) = a_n x^n + a_{n-1} x^n + \cdots + a_1 x + a_0$.

I am thinking of a polynomial all of whose coefficients are non-negative integers. (Perhaps $p(x) = 9x^{56} + 32x^2 + x + 5$, for example, but not $p(x) = \frac{1}{2}x^3 + 7x^2 + 7$ or $p(x) = x^3 - 2x^2 + 8x + 1$.) I won't tell you what my polynomial is, but I will tell you that $p(1) = 9$ and $p(10) = 10332$.

What's $p(53)$ for my polynomial?

SOLVING EQUATIONS
A Too-Quick Historical Overview

QUICK! Solve $ax + b = 0$.

I bet you answered: $x = -b/a$. But who said to solve for x?

Actually French scholar Renè Descartes (1596-1650) did! He proposed that mathematicians use the letters near the end of the alphabet, x, y and z, to denote quantities that are assumed unknown (unknown unknowns) and letters early in the alphabet, a, b, c, … for quantities assumed known (unknown knowns!). Mathematicians thought this a nifty idea and have been following that convention ever since.

LINEAR EQUATIONS, QUADRATIC EQUATIONS
Scholars of very ancient times were able to solve linear equations. The Egyptian Rhind Papyrus, dated ca. 1650 B.C.E., for instance, outlines a method of "false position" for solving equations of the form $ax = b$. Clay tablets, dating ca. 1700 B.C.E., show that Babylonian scholars were solving linear equations and some quadratic equations.

Comment: I should point out that throughout the ages all equations and their solutions were written out in words. The idea of using symbols to represent numbers was very long coming in the history of mathematics. Algebra, as we might recognize it today, wasn't fully developed until the 1500s or so!

The idea of *completing the square* to solve quadratic equations was developed by Greek scholars of 500 B.C.E - via geometry of course. (Look at its name!) The general formula we teach students today:

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

wasn't fully understood and accepted until scholars were comfortable with negative numbers and zero as a number (thank you Brahmagupta, ca. 598 – 665), the full use of irrational numbers as solutions (ca. 1600), and complex numbers as solutions (ca 1750), and, of course, the development of algebraic representation itself! (We expect a great deal of our ninth and tenth graders!)

See the following videos on the mathematics of completing the square (literally!) and deriving the quadratic formula, and more.
www.jamestanton.com/?p=498
www.jamestanton.com/?p=495
www.jamestanton.com/?p=370
www.jamestanton.com?/p=1023

For an overview of the history of algebra and the stories of the people and the mathematical cultures mentioned here see the *Facts on File ENCYCLOPEDIA OF MATHEMATICS* (by yours truly).

Ancient Greek scholars were also interested in solving cubic equations, but struggled with the geometric thinking required to solve them. (They were locked into the geometric mindset.) Finding a general method for solving cubic equations became an infamous problem.

CUBIC EQUATIONS, QUARTIC EQUATIONS and BETRAYAL!

Mathematics was all the rage in 16th-century Italy. Mathematicians were revered and would hold public demonstrations solving mathematical problems and challenging their peers with questions. Patrons sponsored these scholars and mathematicians thus felt the need to keep their problem-solving methods secret. This way they could challenge their peers with questions they themselves could solve but their peers could not. Solving cubic, quartic, quintic, and higher-degree equations become a favored theme.

Comment: I was recently asked: *Why do we classify polynomials by their "degree"? Why the fuss about the highest power of the variable?* The answer is chiefly the history. Our human story of solving equations is locked step-by-step with an increase in the degree of the polynomial at hand. It seems natural then we came to this classification.

Girolamo Cardano (1501-1576) was very interested in general formulas for solving polynomial equations. Curiously, he and his assistant, Lodovico Ferrara, discovered a method for solving quartic equations (degree 4) that relied on solving a cubic equation first - but neither could solve the cubic equation. Frustrating!

In 1539 Cardano caught wind that one of his peers, Niccolò Tartaglia, was solving cubic problems with ease. He visited Tartaglia and urged him to reveal his clever methods. Surprisingly, Tartaglia conceded! He shared his work, but under the strict proviso that Cardano never reveal the methods. (After all, Tartaglia's financial support relied on his ability to continually impress his patron in public competitions.)

Then came betrayal …

Sometime later Cardano received the personal notebook of recently deceased Scipione del Ferro (1465-1526). Del Ferro apparently dabbled in mathematics, recorded all his findings in his notebook, but never shared his ideas. Del Ferro's son-in-law thought the famous Cardano might enjoy the mathematics in the book and so gave it to him. Imagine Cardano's utter surprise upon opening the tome to find all the same methods and techniques Tartaglia had devised for solving cubic equations spelled out in full glorious detail!

After learning that all of Tartaglia's ideas had been discovered by another scholar independently some 30 years earlier, Cardano no longer felt obliged to honor his promise to Tartaglia. He published all the results and methods- the solution to the cubic and his solution to a quartic - in his 1545 treatise *Ars magna*. Although Cardano properly credited Tartaglia, del Ferro, and his assistant Ferrari in the work, Tartaglia was absolutely outraged by this act. His financial patronage was jeopardized and Tartaglia took this as an act of betrayal. A public and bitter dispute ensued.

BACK TO MATHEMATICS: SOME WEIRD SOLUTIONS!

The cubic formula in Cardano's *Ars magna* can lead to some very strange results. For example, in solving equation

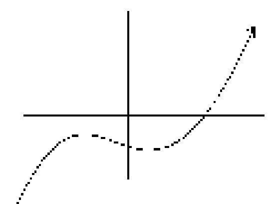
$$x^3 = 6x - 4$$

the method gives the solution:

$$x = \sqrt[3]{-2+\sqrt{-4}} + \sqrt[3]{-2-\sqrt{-4}}\,.$$

(We'll show this in a moment.) Cardano would deem solutions like these as meaningless as they include the square roots of negative quantities. He would reject them.

But Italian mathematician Rafael Bombelli (1526-1573) suggested not being so hasty! After all, every cubic curve must cross the x-axis somewhere and so every cubic curve has at least one real solution. (Including the curve $y = x^3 - 6x + 4$.)



This gave Bombelli the audacity to manipulate an "imaginary" solution like this to discover, in this case, that

$$x = \sqrt[3]{-2+\sqrt{-4}} + \sqrt[3]{-2-\sqrt{-4}}$$

is actually the number $x = 2$ in disguise. And $x = 2$ is a solution to $x^3 = 6x - 4$!

Exercise: Check this! Start by computing $(1+i)^3$ and $(1-i)^3$ to see what the cube roots of $-2 \pm \sqrt{-4}$ are.

Bombelli's observation led mathematicians to change their view of complex solutions and not dismiss them. "Imaginary" solutions might be real and meaningful after all!

THE CUBIC FORMULA: HERE IT IS

There is a reason why Cardano's cubic formula (or any other version of it) is not taught in schools: it is mighty complicated and there are no cute songs for memorizing it! (Why would one want a cute song in the first place?) Let me present the formula here as a series of exercises.

Here is the general cubic equation:

$$x^3 + Ax^2 + Bx + C = 0 .$$

(Assume we have divided through by the any coefficients attached to the x^3 term.)

STEP 1: Put $x = z - \frac{A}{3}$ into this equation to show that it becomes an equation of the form: $z^3 = Dz + E$ for some new constants D and E.

Thus, when solving cubic equations, we can just as well assume that no x^2 term appears. This trick was well known among the Italian mathematicians of the 16th century. But Tartaglia/del Ferro went further with the following truly-inspired idea.

Instead of calling the constants D and E, call them $3p$ and $2q$. This means we need to solve the equation:

$$z^3 = 3pz + 2q$$

STEP 2: Show that if s and t are two numbers that satisfy $st = p$ and $s^3 + t^3 = 2q$, then $z = s + t$ will be a solution to the cubic.

So our job now is to find two numbers s and t satisfying $st = p$ and $s^3 + t^3 = 2q$.

STEP 3: Solve for t in the first equation and substitute the answer into the second to obtains a quadratic equation in s^3. Solve that quadratic equation.

Also write down, and solve, the quadratic equation we would obtain for t^3 if, instead, we solved for s first.

THE FORMULA:
Show that a solution to the cubic equation $z^3 = 3pz + 2q$ is:

$$z = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

[Be careful about the choices of signs here. Recall we must have $s^3 + t^3 = 2q$.]

THAT'S IT! That's the cubic formula!

[Well… we should untangle the meaning of p and q, and rewrite the formula in terms of x, A, B and C. Feel free to do this on your own!]

EXAMPLE: Solve
$$2x^3 + 6x^2 - 6x - 22 = 0 .$$

Answer: Let's first divide through by the leading coefficient of 2 to obtain:

$$x^3 + 3x^2 - 3x - 11 = 0$$

Step 1 says to put $x = z - \frac{A}{3} = z - 1$.

This gives:

$$(z-1)^3 + 3(z-1)^2 - 3(z-1) - 11 = 0$$
$$z^3 - 3z^2 + 3z - 1 + 3z^2 - 6z + 3 - 3z + 3 - 11 = 0$$
$$z^3 - 6z - 6 = 0$$

That is, we need to solve:

$$z^3 = 6z + 6$$

Thus $3p = 6$ and $2q = 6$ giving:

$$p = 2 \text{ and } q = 3$$

Thus:

$$z = \sqrt[3]{3 + \sqrt{9-8}} + \sqrt[3]{3 - \sqrt{9-8}}$$
$$= \sqrt[3]{4} + \sqrt[3]{2}$$

giving the solution:

$$x = z - 1 = \sqrt[3]{4} + \sqrt[3]{2} - 1 .$$

EXAMPLE: Solve $x^3 = 6x - 4$.

Answer: This is already of the correct form for step 1. So to solve it we simply use $3p = 6$ and $2q = -4$ to get: $p = 2$ and $q = -2$. It yields the solution:

$$x = \sqrt[3]{-2+\sqrt{-4}} + \sqrt[3]{-2-\sqrt{-4}},$$

which, as Bombelli would observe, is just $x = 2$ in disguise!

THE QUARTIC FORMULA:

What can I say? It's worse!

For the vehemently enthusiastic algebraists here is a brief outline of a method for solving quartics due to Descarte. (It differs slightly from Cardano's method).

Dividing through by the leading coefficient we may assume we are working with a quartic equation of the form:

$$x^4 + Bx^3 + Cx^2 + Dx + E = 0$$

Substituting $x = y - \frac{B}{4}$ simplifies the equation further to one without a cubic term:

$$y^4 + py^2 + qy + r = 0$$

Make the assumption that this reduced quartic can be factored as follows, for some appropriate choice of number λ, m, and n:

$$y^4 + py^2 + qy + r = \left(y^2 + \lambda y + m\right)\left(y^2 - \lambda y + n\right)$$

Solving $y^4 + py^2 + qy + r = 0$ would then be equivalent to solving

$$\left(y^2 + \lambda y + m\right)\left(y^2 - \lambda y + n\right) = 0$$

which reduces to solving two quadratic equations (which we know how to do).

$$y^2 + \lambda y + m = 0$$
$$y^2 - \lambda y + n = 0$$

So… Are there numbers λ, m, and n for which $y^4 + py^2 + qy + r$ factors as a pair of quadratics: $\left(y^2 + \lambda y + m\right)\left(y^2 - \lambda y + n\right)$?

Expanding brackets and equating coefficients gives the equations:

$$n + m = p + \lambda^2$$
$$n - m = \frac{q}{\lambda}$$
$$nm = r$$

If we can solve λ, m, and n, we indeed have the desired factoring.

Summing the first two equations gives $n = \dfrac{p + \lambda^2 + \frac{q}{\lambda}}{2}$; subtracting them yields $m = \dfrac{p + \lambda^2 - \frac{q}{\lambda}}{2}$; and substituting into the third equation yields, after some algebraic work, a cubic equation solely in terms of λ^2:

$$(\lambda^2)^3 + 2p\left(\lambda^2\right)^2 + \left(p^2 - 4r\right)\left(\lambda^2\right) - q^2 = 0$$

Cardano's cubic formula can now be used to solve for λ^2, and hence for λ and then for m, and n.

Now go back and solve those two quadratic equations for y, and then recall $x = y - \frac{B}{4}$.

EASY! (Hmm.)

QUINTICS AND BEYOND!

During the 1600s and 1700s, there was great eagerness to find a similar formula for the solution to the quintic (degree-5 equation). The great Swiss mathematician Leonhard Euler (1707-1783) attempted to find such a formula, but failed. He suspected that the task might be impossible

Comment: This would be mighty odd! Why would there be formulas for solving degree 1 (linear), degree 2 (quadratic), degree 3 (cubic) and degree 4 (quartic) equations, but suddenly not for degree 5 equations?

In a series of papers published between the years 1803 and 1813, Italian mathematician Paolo Ruffini developed a number of algebraic results that strongly suggested that there can be no procedure for solving a general fifth- or higher-degree equation in a finite number of algebraic steps.

And this surprising thought was indeed proven correct a few years later by Norwegian mathematician Niels Henrik Abel (1802-1829). Thus—although there is the quadratic formula for solving degree-2 equations, a formula for solving degree-3 equations, and another for degree-4 equations—there will never be general formula for solving all equations of degree-5 equations or all degree six equations, or all degree seven equations, and so on!

This was a surprising and shocking end to an almost 4000 year-long mathematical quest!

Comment: Of course some specific degree-five equations can be solved algebraically with formulas. (Equations of the form $x^5 - a = 0$, for instance, have solutions $x = \sqrt[5]{a}$.) In 1831, French mathematician Évariste Galois completely classified those equations that can be so solved with algebraic formulas. This work gave rise to a whole new branch of mathematics today called group theory.

RESEARCH CORNER:

The opening puzzler is intriguing!

It is very surprising that with just two pieces of input-output information one can completely determine a polynomial known to have non-negative integer coefficients.

For example, from $p(1) = 9$ and $p(10) = 10332$, it must be that $p(x) = x^4 + 3x^2 + 3x + 2$.

REASON:
Write $p(x) = a_n x^n + \cdots + a_1 x + a_0$.
$p(1) = a_n + a_{n-1} + \cdots + a_1 + a_0 = 9$ shows us that each a_i is a digit between 0 and 9.

$$p(10) = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10 + a_0 = 10332$$

shows us that $p(x) = x^4 + 3x^2 + 3x + 2$ since there is only one way to write 10332 in base 10!

IN GENERAL: *For a polynomial $p(x)$ with non-negative integer coefficients, explain why knowing the values $p(1)$ and $p(p(1)+1)$ is enough to determine $p(x)$.*

Research: Suppose I tell you that $p(x)$ has integer coefficients, one of which is negative and the rest are non-negative. Could you determine what the polynomial is from a finite sequence of input-output questions? How many do you need?

What if I told you that all but two of the integer coefficients were non-negative?

tanton.math@gmail.com