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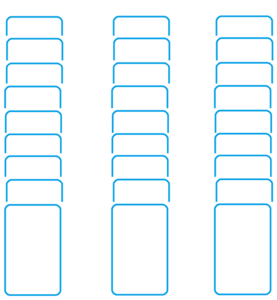
WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

SEPTEMBER 2020

I was recently learned of the famous *27 Card Trick* and discovered Matt Parker's popular 2012 Numberphyle video of it here. Alas, even after watching the explanation, I still didn't "get it." I then spent an afternoon pondering and puzzling.



This month I share my thoughts on the mathematics I think is lurking behind the scenes. Let's start with a simpler version of the trick in the opening puzzler.

Martin Gardner discusses the puzzle and its history in *Mathematics, Magic and Mystery* (Dover, 1956).

THIS MONTHS' PUZZLER:

Deal out 27 face-up cards on a table-top, three columns of nine. Do this by dealing out a row of three cards, and then a second row of three cards overlapping these, and so on. Make sure the numbers and suits of all 27 cards are visible.

Ask a friend to secretly select a card and memorize what it is. Then ask her to indicate the column in which it lies.

Now scoop up each column in turn to make a single pile of 27 facedown cards. Just be sure to pick up the indicated column second so that the middle nine cards of the stack of 27 cards are the cards of the indicated column.

Now lay out the cards again, nine rows of three, as before. Have your friend indicate again in which column the secret card lies and scoop up the columns again, again making the indicated column of cards is in the middle of the stack.

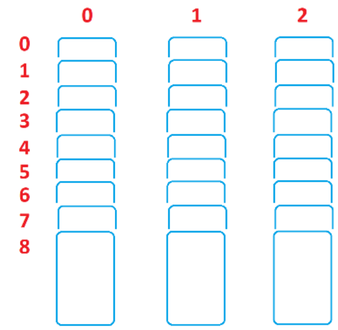
Deal out the cards one more time, nine rows of three, as before. This time when your friend indicates the column containing her secret card, to her astonishment, you announce what the card is: It is simply the middle card of that column.

Can you explain why this is sure to be so?

THE MATHEMATICS OF LAYING OUT CARDS

Our patten of laying out cards focuses on groups of three and it is likely we want to keep track of counts that are 0 more, 1 more, or 2 more than a multiple of three.

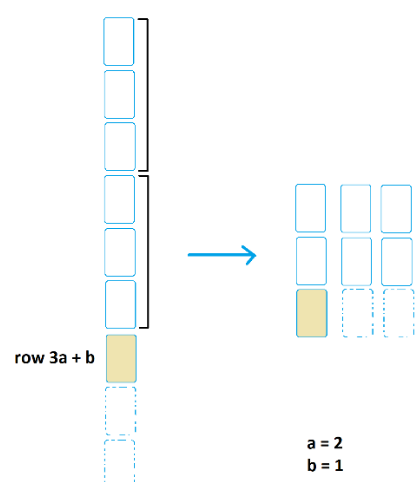
My point is that intuition tells me that it might be good to have the number 0 in our count of things.



So, let's say that the card in the top left corner of a layout of nine-by-three cards is in row 0 and column 0, and the card in the bottom right corner is in row 8, column 2. That is, let's number the rows 0 to 8, and the columns 0, 1, and 2.

Suppose your friend's secret card is in row $3a+b$, column c, with each of the numbers a, b, c either 0, 1, or 2. You, of course, do not know this and your friend only shows you the column in which her secret card sits. That is, she reveals to you the value c, nothing more.

Ignoring the other two columns for a moment, if you pick up that one indicated column and, with the nine cards face down, deal out three rows of three, the secret card will appear in row a, column b of that three-by-three array. This follows as any set of $3a+b$ cards is composed of a groups of three (and each group of three is laid out in a row so that makes for a rows) and b more cards.



Now consider picking up all three columns to make a stack of 27 cards all face down which we'll deal out in rows of three. If we put the indicated column in the top of the stack as the top third of the 27 cards, then the secret card will be row a, column b of the nine-by-three array. If we put the indicated column in the middle of the stack as the middle third of the 27 cards, then the secret card will be row $3+a$, column b of the nine-by-three array. If we put the indicated column in at bottom of the stack as the bottom third of the 27 cards, then the secret card will be row $6+a$, column b of the nine-by-three array.

Let A be the number 0 (for top), 1 (for middle), or 2 (for bottom) to indicate which third of the pile of 27 cards we choose to put the indicated stack. (The opening puzzle has $A=1$.)

We have:

Suppose that the secret card is in position

row $3a+b$
column c.

in the nine-by-three array.

And suppose when we pick up the three columns, we put the column containing the secret card as the A th third of the stack 27 facedown cards.

Then when we lay out a new nine-by-three array, the secret card will be in position

row $3A+a$
column b.

Heads up: Look at how new placement of the symbols changed.

Let's repeat this process, but this time we'll put the indicated column as the B th third of the stack of the 27 cards. By the result we just established, the secret card will now be in position

row $3B+A$
column a.

And let's do it a third time, now putting the indicated column as the C th third of the stack of the 27 cards. The secret card will now be in position

row $3C+B$
column A

in the nine-by-three array of cards.

And look! We get to choose the values A, B, and C and so we have complete control of the location of the secret card after these three moves without ever being privy to its initial row number.

The opening puzzle chooses $A=B=C=1$ and so at the end of this process the card will be in row 4, column 1, right in the middle of the array. (Or if you don't deal out the cards a third time, you know that the secret card is in row 4 of the indicated column in the previous step.)

THE FULL TRICK

At the start of the trick ask our friend to select a card from a pile of 27 cards and to also think of a number between 0 and 26. (Well, most people think that is weird, so ask them to select a number between 1 and 27 and you then secretly subtract one from it.) Call this number between 0 and 26 N and mentally find the values A, B, and C such that

$$N = 9C + 3B + A.$$

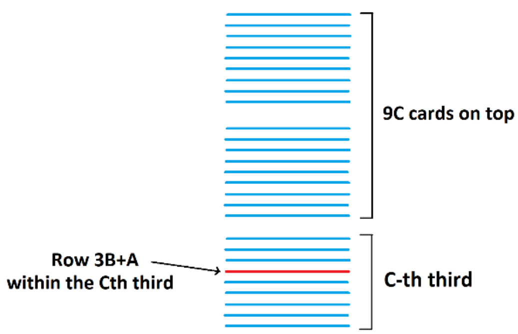
You will work with the values A, B, and C.

Lay out the 27 cards in a nine-by-three array. Have your friend indicate the column of the secret card and scoop up the three columns with the selected column in the Ath third of stack of 27 facedown cards.

Deal out the 27 cards in a nine-by-three array again. Have your friend indicate the column of the secret card and scoop up the three columns with the selected column in the Bth third of stack of 27 facedown cards.

Deal out the 27 cards in a nine-by-three array one more time. Have your friend indicate the column of the secret card and scoop up the three columns with the selected column in the Cth third of the stack of 27 facedown cards.

But don't deal out the cards again. Stop here.



In the previous layout we know the secret card was in row $3B + A$ of the indicated column. By placing the indicated column as the Cth third in the stack of 27 facedown cards we have, in fact, placed the secret card in position $9C + 3B + A$ in the stack.

If we start flicking the cards off the top of the stack, counting 0, 1, 2, and so on, and turn over the card of number $N = 9A + 3B + C$, one less than your friend's chosen number, we'll reveal the secret card.

But this is not the number your friend chose! So, instead, start flicking off the cards from the top of the pile starting counting with 1, 2, 3, … and then the card at your friend's number will be their secret card!

This is the trick that impresses.

Comment: No doubt this essay is still a bit hard to parse. Try working through it again with 27 cards in hand. It will help make sense of the mathematics.

RESEARCH CORNER

1. What's the 25 card trick making use of representing secret numbers as $5A + B$?

2. What is the general b^r card trick represent secret numbers in base b?

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