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WOWZA! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

October 2018

SPECIAL EDITION!

To honor the second annual

GLOBAL MATH WEEK

Oct 10 - 17

www.globalmathproject.org

we present here a new take on *Exploding Dots*. The *Julia Robinson Math Festival* folk are using the ideas presented here for a brand new booklet to share with the world (http://jrmf.org/booklets/), and I've shared this material as a brand new experience – Experience 11 – on my website, replete with videos! (www.gdaymath.com/courses/exploding-dots/).

This content is perfect for sharing with a mixed audience of students – some who have seen Exploding Dots before and some who have not. (Maybe go to my website and view the videos there on this material with your students.)

I hope you enjoy this fun grape codes!

SETTING THE SCENE: GRAPE CODES

Consider a row of dishes extending as far to the left as ever one desires, each labeled with a power of two, in order, starting from the right. In the picture I have six dishes.

Question 1: *If I had ten dishes what would be the label of the leftmost dish?*

I put grapes in my dishes and when I do each grape has value given by the label of the dish in which it sits. For example, three grapes in the 8 dish and two in the 1 dish together have a total value of $8+8+8+1+1=26$. I will write $3|0|0|2$ as a code for twenty-six. (I'll ignore all leading zeros, that is, I won't record the empty dishes to the left of the leftmost non-empty dish.) Other "grape codes" for twenty-six are possible.

3|0|1|0

6|0|2

1|1|0|1|0

26

Question 2: *There happen to be a total of 114 different grape codes for the number twenty-six. That is, there are 114 different ways to represent the number twenty-six with grapes in dishes. The code $3|0|1|0$ represents the number twenty-six with just four grapes. The code $6|0|2$ uses eight grapes.*

Of all 114 codes for twenty-six, is "26" the code that uses the most number of grapes? Is $1|1|0|1|0$ the code that uses the least number of grapes? How do you know?

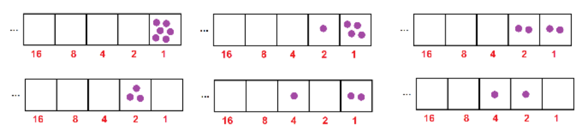
Are there two different codes for twenty-six that use the same count of grapes? Are there five different codes that use the same count of grapes?

Question 3: *Here are the first few numbers that have codes using only two grapes.*

2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 18, 20, 24, 32, 33, 34, 36, …

What is the 50th number in this list?

Question 4: *There 6 different grape codes for the number six.*



a) *Show that there are also 6 grape codes for the number seven. Actually draw diagrams for each of the codes.*

b) *Is it true in general that the count of grape codes for an odd number is sure to equal the count of grape codes for the even number just before it?*

The table shows the number of different grape codes for the first few even numbers.

Number	2	4	6	8	10	12	14	16	26
# of grape codes	2	4	6	10	14	?	?	?	114

c) *Fill in the three missing entries. Care to find a few more entries?*

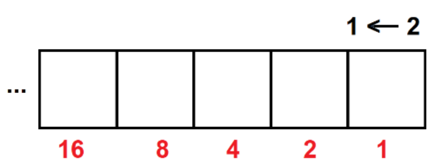
d) *Is there a pattern to the sequence of numbers you are finding:* $2, 4, 6, 10, 14, \ldots\ldots$
(Can you be sure any patterns you see are genuine?)

Question 5: *A code for a number with at most one grape in each dish is called a* binary code *for that number. For instance,* $1|1|0|1|0$ *is a binary code for the number twenty-six and* $1|1|0$ *is a binary code for the number six.*

a) *Find a binary code for the number fifty.*
b) *Is every positive integer sure to have a binary code?*
(We'll actually answer this question soon!)
c) ***HARD CHALLENGE:*** *Could a positive integer have two different binary codes?*

THE $1 \leftarrow 2$ MACHINE

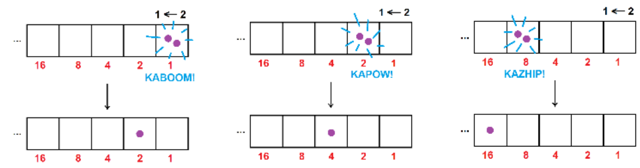
In the story of *Exploding Dots* of the [Global Math Project]() our row of dishes is simply a "two-one machine," written $1 \leftarrow 2$.



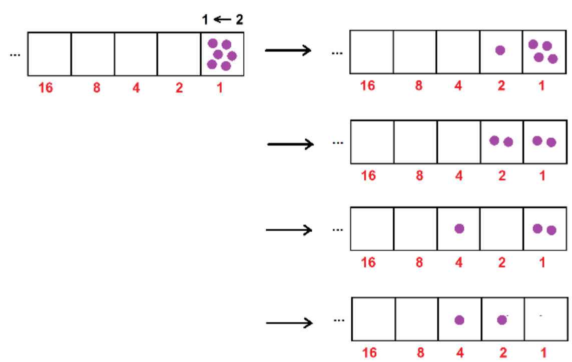
One puts in dots (or grapes) in the rightmost box and let them "explode" in the following way:

Whenever there are two dots in a box, any box, they explode and disappear - KAPOW! – to be replaced by one dot, one box to the left.

And, indeed, two dots in any one box have the same combined value as one dot just to their left.



In this way, placing a number of dots in the rightmost gives a representation of that number with at most one dot in each box. This proves that every positive integer has at least one binary code. For example, placing 6 dots into the machine eventually gives the binary code $1|1|0$ for the number 6.



Question 6: *Which number has binary code* $1|0|1|1$?
Which number has binary code $1|0|1|1|0$ *and which has code* $1|0|1|1|1$?

Question 7: a) *Find the binary codes of the first twenty positive integers. What do you notice about the codes of the even numbers? The codes of the odd numbers?*

b) Anouk says she invented a divisibility rule for the number 4:

A number is divisible 4 precisely when its binary code ends with two zeros.

Do you agree with her rule?

c) Is there a divisibility rule for the number 3 *based on the binary code of numbers?*

Question 8:

a) Aba has a curious technique for finding the binary code of a number. She writes the number at the right of a page and halves it, writing the answer one place to its left, ignoring any fractions if the number was odd. She then repeats this process until she gets the number 1. Then she writes 1 under each odd number she sees and 0 under each even number. The result is the binary code of the original number.

Here's her work for computing the binary code of 22.

1	2	5	11	22
1	0	1	1	0

Why does her technique work?

(HINT: Put 22 dots in a $1 \leftarrow 2$ machine and watch what happens.)

b) **FOLLOW-ON CHALLENGE:**

Here's a fun way to compute the product of two numbers, say, 22×13. Write the two numbers at the head of two columns, halve the left number (ignoring in fractions) and double the right number, and repeat until the number 1 appears. Then cross out all the rows that have an even number on the left, and add all the numbers on the right that survive. That sum is the answer to the original product!

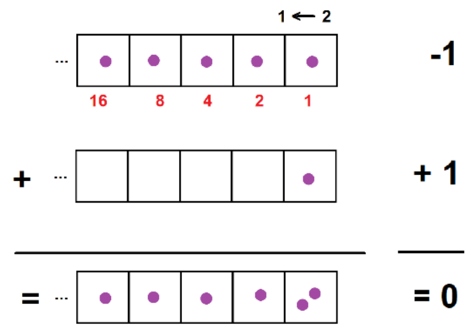
~~22 x 13~~
11 x 26
5 x 52
~~2 x 104~~
1 x 208

286

Why does this method work?

(Hint: $208 + 52 + 26 = 16 \times 13 + 4 \times 13 + 2 \times 13$ and $22 = 16 + 4 + 2$.)

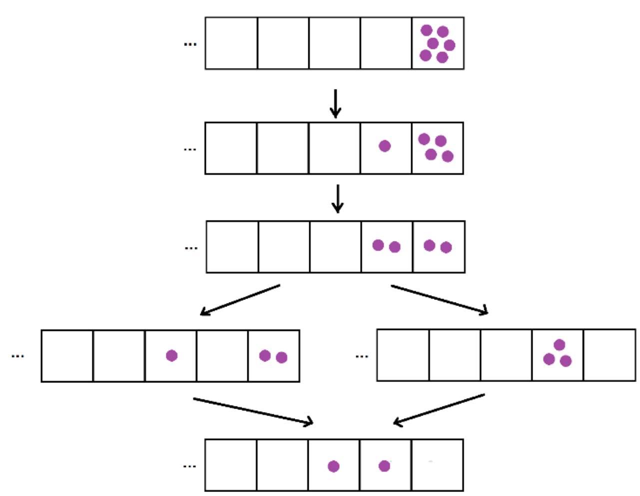
Question 9: *Allistaire suggested that the binary code of -1 should be $\cdots 1|1|1|1|1|1|1|1$ (that is, an infinitely long string of ones going infinitely far to the left). He argued that adding one more dot to a $1 \leftarrow 2$ machine with a dot in each box produces, after explosions, an empty diagram: zero.*



Do you agree?

GRAPE CODES AND BINARY CODES

The following diagram shows all the choices one can make when performing explosions on 6 dots to lead to the binary code $1|1|0$ for the number 6. The diagram also shows all 6 ways we can represent 6 with grapes!



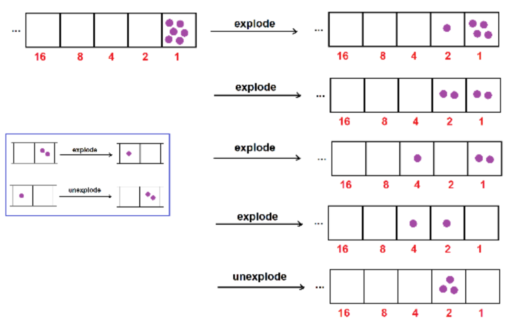
Question 10: *a) Draw an analogous diagram for 12 dots placed in a $1 \leftarrow 2$ machine. Show all the choices one can make*

for explosions and show that all paths lead to the same final binary code $1|1|0|0$.

b) There are 20 *ways to represent the number* 12 *with grapes in dishes. Do all* 20 *grape codes appear in your diagram? Do all paths lead to the same binary code of* 12*?*

c) In general, when one draws a diagram of all possible explosions for N *dots placed in a* $1 \leftarrow 2$ *machine, is the diagram sure to contain all the possible codes of* N *with grapes? Do all paths lead to the same final binary code for* N*?*

Question 11: *Starting with* 6 *dots in a* $1 \leftarrow 2$ *machine, one can perform a sequence of five explosions and "unexplosions" that produces all* 6 *codes for* 6 *in terms of grapes.*



It turns out that for any positive integer N *there is a sequence of explosions and unexplosions one can perform—starting with* N *dots in the rightmost box of a* $1 \leftarrow 2$ *machine—to pass through all the possible grape codes of* N *without repeating a code.*

Starting with 12 *dots in the* $1 \leftarrow 2$ *machine, can you find the a sequence of* 19 *explosions and unexplosions that takes one through all* 20 *possible codes for* 12 *in terms of grapes?*

Comment: The 2018 ARML power question also explores these questions about codes for numbers, but not in the language of grapes or of *Exploding Dots*.

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