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WHAT COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

OCTOBER 2015

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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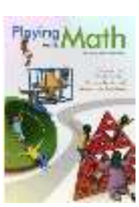
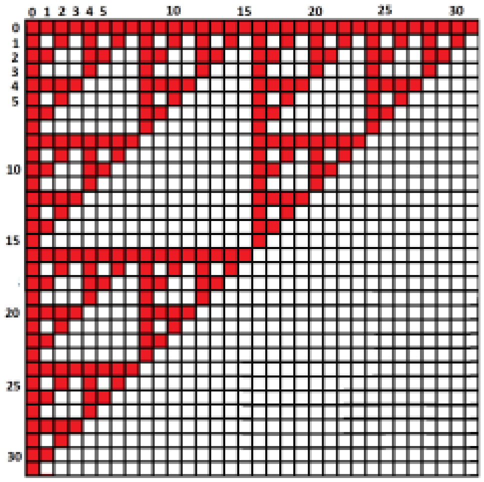


A New Book! *Playing with Math: Stories from Math Circles, Homeschoolers, and Passionate Teachers*, edited by Sue VanHattum.

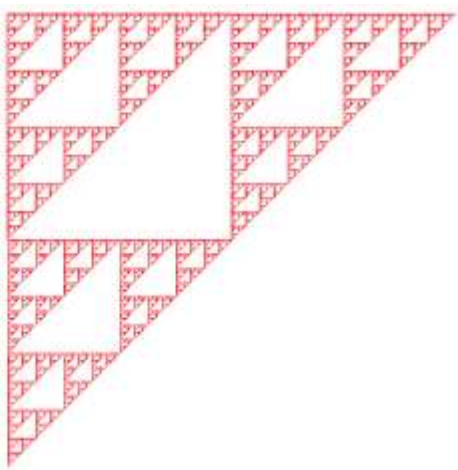
A wonderful collection of puzzles, games, and activities, each following a chapter about how people are sharing their joyous encounters with math.
naturalmath.com/playingwithmath/

OPENING PUZZLE: Call two natural numbers a and b disjoint if the 1s in the binary representation of a appear in different positions from the 1s in the binary representation of b. For example, $26 = 11010_2$ and $8 = 100_2$ are disjoint, as are $34 = 100010_2$ and $21 = 10101_2$. Every number is disjoint with zero.

Here's a table of pairs with the (a,b)-cell colored red if a and b are disjoint.



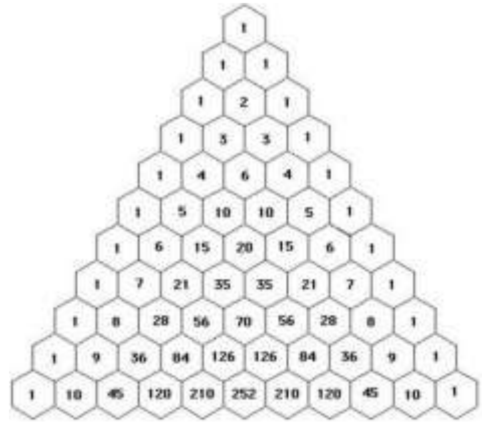
The self-replicating structure persists if we grow the table.



Explain the structure we see.

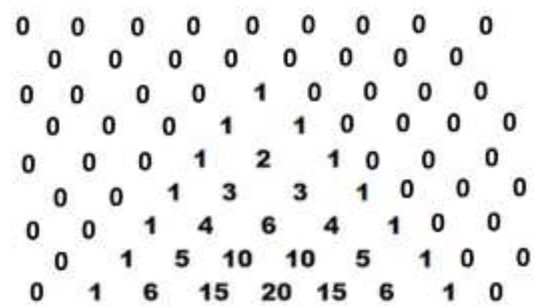
PASCAL'S TRIANGLE

Pascal's famous triangle is constructed as follows: *Place a* 1 *at the apex of a triangle and* 1*s along its two sides. Each interior entry is then the sum of the two entries just above it.*



The first ten rows of Pascal's triangle.

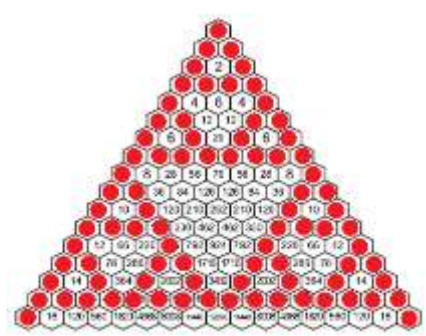
Actually, the rule described generates the sides of the triangle too if we imagine the array as sitting in a sea of zeros: all but the apex 1 is the sum of the two numbers just above it.



A 1 *placed anywhere in a sea of zeros generates Pascal's triangle via the sum rule.*

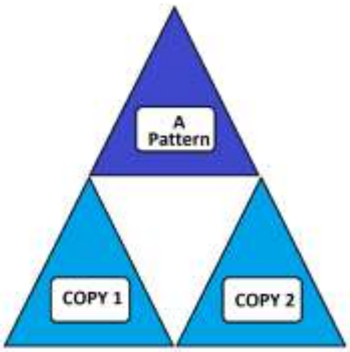
SELF-REPLICATING STRUCTURE IN PASCAL'S TRIANGLE

Color red the cells of Pascal's triangle that contain an odd number. We see the same self-repeating triangular structure (tilted 45 degrees) as that of the opening puzzle!



Let's describe that structure carefully. It seems:

- Rows $1, 2, 4, 8, 16, \ldots$, the power of two numbered rows of Pascal's triangle, are fully red. (That is, these rows of the triangle contain only odd entries.)
- After each power-of-two row, two copies of the structure developed in the triangle thus far appear, one left and one right. These copies "touch" at the next power-of-two row.



The self-replicating structure.

It is possible to explain this structure.

Since we are interested only in the evenness and oddness of numbers, we can replace each even number in Pascal's triangle with a 0 and each odd number with a 1. (That is, replace each entry with its remainder upon division by two.) And since

$$even + even = even$$
$$even + odd = odd$$
$$odd + even = odd$$
$$odd + odd = even$$

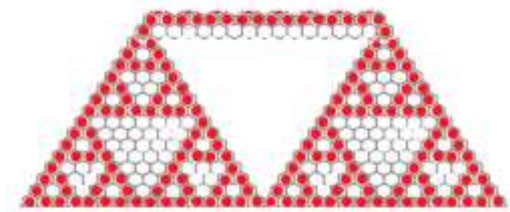
we can follow the arithmetic rules

$$0+0=0$$
$$1+0=1$$
$$0+1=1$$
$$1+1=0.$$

Each 0 (each even number) in the triangle is the sum of either two 0s or two 1s just above it, and each 1 (each odd number) is the sum of a 0 and a 1 just above it.

We can check directly that rows $1, 2, 4, 8$, and 16 are indeed all 1s (all odd entries). The rules of our arithmetic show that row 17 must have a 1 at its start, a 1 at its end, and be zero everywhere else.

Each of the 1s in row 17 is sitting in its own sea of zeros, and so will generate its own copy of Pascal's triangle, until the copies touch and begin to interfere with each other. Each copy can proceed 16 rows without interference and the two triangles will just touch on row 32. As the sixteenth row of the triangle is all 1s, so is row 32 of the triangle.



Now we are in a cycle: Row 33 will begin and end with a 1. These 1s each sit in a sea of zeros and so will generate its own fresh copies of Pascal's triangle. Each copy grows without interference for 32 lines. But on row 64 they touch, with two adjacent rows of 1s. Thus the 64th row is nothing but 1s and off we go again!

THE DISJOINT TRIANGLE

Let's introduce the notation

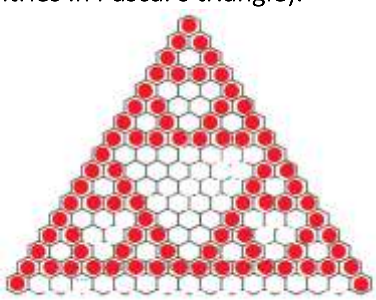
$$\left\langle \begin{matrix} a \\ b \end{matrix} \right\rangle = \begin{cases} 1 & \text{if } a \text{ and } b \text{ are disjoint} \\ 0 & \text{otherwise} \end{cases}$$

and display the array of these values as a triangle rather than the rectangular array of the opening puzzler. (Again, just tilt the opening rectangular array 45°.)

$$\left\langle \begin{matrix}0\\0\end{matrix} \right\rangle$$
$$\left\langle \begin{matrix}0\\1\end{matrix} \right\rangle \quad \left\langle \begin{matrix}1\\0\end{matrix} \right\rangle$$
$$\left\langle \begin{matrix}0\\2\end{matrix} \right\rangle \quad \left\langle \begin{matrix}1\\1\end{matrix} \right\rangle \quad \left\langle \begin{matrix}2\\0\end{matrix} \right\rangle$$
$$\left\langle \begin{matrix}0\\3\end{matrix} \right\rangle \quad \left\langle \begin{matrix}1\\2\end{matrix} \right\rangle \quad \left\langle \begin{matrix}2\\1\end{matrix} \right\rangle \quad \left\langle \begin{matrix}3\\0\end{matrix} \right\rangle$$

Let's call this triangle of 0 s and 1s the disjoint triangle.

The puzzler asks us to explain why the disjoint triangle is identical to Pascal's $0/1$ triangle (that is, to the pattern of even and odd entries in Pascal's triangle).



One way to establish this would be to prove that the disjoint triangle entries follow exactly the same rule Pascal's triangle: each entry (except for the apex) is the sum of the two entries above it. Try it!

CHALLENGE 1: *Prove that*

$$\left\langle \begin{matrix}a\\b\end{matrix} \right\rangle = \left\langle \begin{matrix}a-1\\b\end{matrix} \right\rangle + \left\langle \begin{matrix}a\\b-1\end{matrix} \right\rangle$$

if we follow the rules

$$0+0=0$$
$$1+0=1$$
$$0+1=1$$
$$1+1=0.$$

That is, prove that each entry in the disjoint triangle is the sum of the two entries just above it.

Another approach would be explain a direct connection between each entry of the disjoint triangle and its matching entry in Pascal's triangle.

A general entry of Pascal's triangle is $\frac{(a+b)!}{a!b!}$. What is the connection between this value and the value of $\left\langle \begin{matrix}a\\b\end{matrix} \right\rangle$?

CHALLENGE 2: *Prove that two numbers a and b are disjoint precisely when $\frac{(a+b)!}{a!b!}$ is odd.*

EXPLORING CHALLENGE 2:

The integer $\frac{(a+b)!}{a!b!}$ is odd only if the count of 2 s that appear in the prime factorization of $(a+b)!$ matches the count of 2 s that appear in the prime factorization of $a!$ plus the count that appear in that of $b!$ (We want all the 2 s to cancel.)

Now, the prime factorization of a factorial, $N! = 1\times2\times3\times4\times5\times6\times7\times8\times\cdots\times N$ for some natural number N, say, has one 2 coming from each the numbers $2, 4, 6, 8, 10$, and so on up to the largest multiple of two less than or equal to N, and an extra 2 from each multiple of four, 4, 8, 12, … less than or equal to N, and an additional 2 from each multiple of 8 less than equal to N, and yet another 2 from each multiple of 16, and so on.

EXERCISE: Prove that there are $\left\lfloor \frac{N}{k} \right\rfloor$ multiples of k among the numbers $1, 2, 3, 4, 5, \ldots, N$. Here $\lfloor x \rfloor$ represents largest integer less than or equal to x. (Hint: Write $N = km + r$.)

According to this exercise there are $\left\lfloor \frac{N}{2} \right\rfloor$ multiples of 2 we need to consider, $\left\lfloor \frac{N}{4} \right\rfloor$

multiples of 4, $\left\lfloor \frac{N}{8} \right\rfloor$ multiples of 8, and so on. This means that the prime factorization of $N!$ has

$$\left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{4} \right\rfloor + \left\lfloor \frac{N}{8} \right\rfloor + \left\lfloor \frac{N}{16} \right\rfloor + \cdots$$

twos in it. (Eventually the terms in this sum become zero, so this is really just a finite sum of numbers.)

To check our work, notice that

$$8! = 1\times(2)\times 3\times(2\times 2)\times 5 \times(2\times 3)\times 7\times(2\times 2\times 2)$$

has one two from each of 2, 4, 6, 8, another two from each of 4, 8, a third two from 8. There are

$$\left\lfloor \frac{8}{2} \right\rfloor + \left\lfloor \frac{8}{4} \right\rfloor + \left\lfloor \frac{8}{8} \right\rfloor + \left\lfloor \frac{8}{16} \right\rfloor + \cdots$$
$$= 4+2+1+0+0+0+\cdots$$
$$= 7$$

twos in the prime factorization of $8!$.

To examine $\frac{(a+b)!}{a!b!}$, we see that there are

$$\left\lfloor \frac{a+b}{2} \right\rfloor + \left\lfloor \frac{a+b}{4} \right\rfloor + \left\lfloor \frac{a+b}{8} \right\rfloor + \cdots$$

twos in $(a+b)!$, and

$$\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{a}{8} \right\rfloor + \cdots$$
$$\left\lfloor \frac{b}{2} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor + \left\lfloor \frac{b}{8} \right\rfloor + \cdots$$

twos in $a!$ and $b!$, respectively.

For $\frac{(a+b)!}{a!b!}$ to be odd, we need

$$\left\lfloor \frac{a+b}{2} \right\rfloor + \left\lfloor \frac{a+b}{4} \right\rfloor + \left\lfloor \frac{a+b}{8} \right\rfloor + \cdots$$
$$= \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{a}{8} \right\rfloor + \cdots$$
$$+ \left\lfloor \frac{b}{2} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor + \left\lfloor \frac{b}{8} \right\rfloor + \cdots.$$

Although there is no reason to think that individual terms in the sum should align, one does wonder:

When does $\left\lfloor \frac{a+b}{2^k} \right\rfloor = \left\lfloor \frac{a}{2^k} \right\rfloor + \left\lfloor \frac{b}{2^k} \right\rfloor$*?*

FOCUSING ON INTEGER PARTS

Write each of a and b in base two:

$$a = a_0 + a_1\cdot 2 + a_2\cdot 2^2 + a_3\cdot 2^3 + \cdots$$
$$b = b_0 + b_1\cdot 2 + b_2\cdot 2^2 + b_3\cdot 2^3 + \cdots$$

Here each a_i and b_i is either 0 or 1.

Then $a+b$ equals

$$(a_0+b_0) + (a_1+b_1)\cdot 2 + (a_2+b_2)\cdot 2^2 + \cdots.$$

But this might not be the base two representation of $a+b$: If $a_9 = 1$ and $b_9 = 1$, say, then $a_9 + b_9 = 2$ and is not 0 or 1. (In base two arithmetic, one then "carries a one" at that 9th place and adds a one to the 10th place.) So the base-two representation of $a+b$ might be delicate.

Now

$$\frac{a}{2^k} = \frac{a_0}{2^k} + \frac{a_1}{2^{k-1}} + \cdots + \frac{a_{k-1}}{2} + a_k + a_{k+1}\cdot 2 + a_{k+2}\cdot 2^2 + \cdots$$

and we see

$$\left\lfloor \frac{a}{2^k} \right\rfloor = a_k + a_{k+1}\cdot 2 + a_{k+2}\cdot 2^2 + \cdots.$$

Similarly

$$\left\lfloor \frac{b}{2^k} \right\rfloor = b_k + b_{k+1}\cdot 2 + b_{k+2}\cdot 2^2 + \cdots.$$

Also,

$$\frac{a+b}{2^k} = \frac{a_0+b_0}{2^k} + \cdots + \frac{a_{k-1}+b_{k-1}}{2} + (a_k+b_k) + (a_{k+1}+b_{k+1})\cdot 2 + \cdots$$

and so

$$\left\lfloor \frac{a+b}{2^k} \right\rfloor = (a_k+b_k) + (a_{k+1}+b_{k+1})\cdot 2 + \cdots = \left\lfloor \frac{a}{2^k} \right\rfloor + \left\lfloor \frac{b}{2^k} \right\rfloor$$

if a_{k-1} and b_{k-1} aren't both 1.
If they are both 1, then

$$\left\lfloor \frac{a+b}{2^k} \right\rfloor = \frac{a_{k-1}+b_{k-1}}{2} + (a_k+b_k) + (a_{k+1}+b_{k+1})\cdot 2 + \cdots = 1 + \left\lfloor \frac{a}{2^k} \right\rfloor + \left\lfloor \frac{b}{2^k} \right\rfloor$$

This means,

$$\left\lfloor \frac{a+b}{2} \right\rfloor + \left\lfloor \frac{a+b}{4} \right\rfloor + \left\lfloor \frac{a+b}{8} \right\rfloor + \cdots$$

equals

$$\left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a}{4} \right\rfloor + \left\lfloor \frac{a}{8} \right\rfloor + \cdots + \left\lfloor \frac{b}{2} \right\rfloor + \left\lfloor \frac{b}{4} \right\rfloor + \left\lfloor \frac{b}{8} \right\rfloor + \cdots.$$

only if we never have $a_{k-1} = b_{k-1} = 1$ for any k. That is, $\frac{(a+b)!}{a!b!}$ is odd, only if the binary representations of a and b never have a 1 in the same position. That is, $\frac{(a+b)!}{a!b!}$ is odd precisely when a and b are disjoint.

This establishes challenge 2.

We have now proved that that the disjoint triangle is identical to the odd/even version of Pascal's triangle.

Comment: We've actually proved more. We showed that

$$\left(\left\lfloor \frac{a+b}{2} \right\rfloor - \left\lfloor \frac{a}{2} \right\rfloor - \left\lfloor \frac{b}{2} \right\rfloor\right) + \left(\left\lfloor \frac{a+b}{4} \right\rfloor - \left\lfloor \frac{a}{4} \right\rfloor - \left\lfloor \frac{b}{4} \right\rfloor\right) + \left(\left\lfloor \frac{a+b}{8} \right\rfloor - \left\lfloor \frac{a}{8} \right\rfloor - \left\lfloor \frac{b}{8} \right\rfloor\right) + \cdots$$

is the number of 2 s in the prime factorization of $\frac{(a+b)!}{a!b!}$, and that this equals the number of times that a and b have a 1 in the same column of their binary representations.

German mathematician Ernst Kummer noticed this too in 1852 and phrased matters this way:

The number of 2 s in the prime factorization of $\frac{(a+b)!}{a!b!}$ equals the number of carries one must perform when adding a and b in base 2.

This work actually holds for any prime p :

The number of p *s in the prime factorization of* $\frac{(a+b)!}{a!b!}$ *equals the number of carries one must perform when adding* a *and* b *in base* p *.*

This link between the primes in the prime factorization of the (a,b) entry of Pascal's triangle and the number of carries one must perform in computing $a+b$ in a prime base is today called Kummer's Theorem.

Challenge: Think through the work of this essay for a prime p different from 2. Establish Kummer's general result.

RESEARCH CORNER

Replace each entry in Pascal's triangle with the number of 2 s in the prime factorization of that entry.

```
      0
     0 0
    0 1 0
   0 0 0 0
  0 1 1 1 0
 0 0 1 1 0 0
0 1 0 2 0 1 0
```

This essay shows that that the zeros in this array (the odd entries) follow a self-replicating triangle structure.

What structure do the 1s follow? The 2 s? The 3s? The 4 s? …

Repeat this work for primes other than 2.

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