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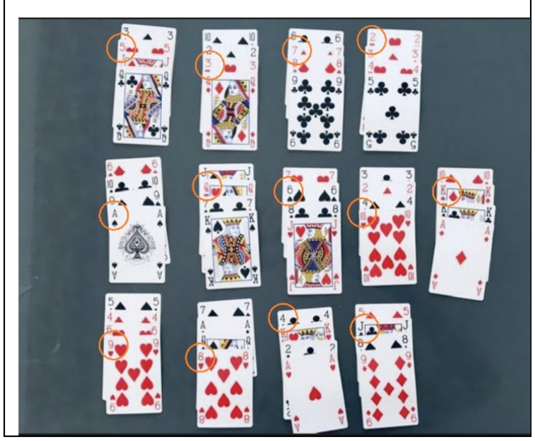
GLOBAL MATH PROJECT
Uplifting Mathematics for All

WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

NOVEMBER 2020

THIS MONTHS' PUZZLER:



1. Shuffle a deck of 52 cards and split the deck into two piles of 26 cards. It is plainly possible, no matter the split, to pull out a red card from one pile and a black card from the other. But can you do it again from what remains—pull out one card of each color from two different piles—and again, and again, fifty-two times in a row?

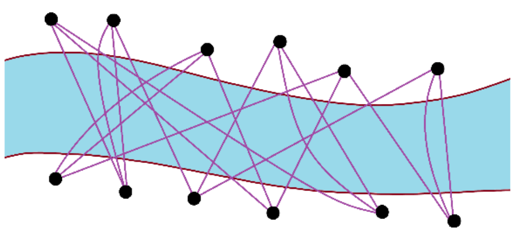
2. Shuffle a deck of 52 cards and split the deck into four piles of 13 cards. Is it possible, no matter the split, to pull out four cards, one of each suit, from four different piles? If so, could you do it a

second time with what remains? A third time? Thirteen times in a row?

3. Shuffle a deck of 52 cards and split the deck into thirteen piles of 4 cards. Is it possible, no matter the split, to pull thirteen cards, one of each face value, from thirteen different piles? If so, could you do it a second time from what remains? A third time? A final fourth time?

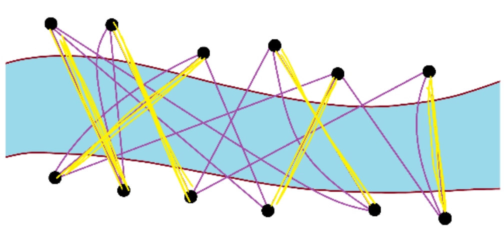
ROPES ACROSS A RIVER

On each side of a river stand six people, each holding three ropes that cross the river to one or more people on the opposite side.



Is it possible for each person to let go of two ropes to leave each person holding just one rope with another person across the river?

In this picture we see the answer is YES! Have folk just hold the yellow ropes shown and drop all other ropes.



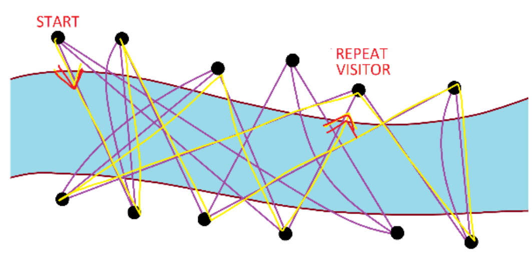
But are <u>all</u> pictures of 6 people each holding 3 ropes amendable to such solutions? Are the numbers "6" and "3" special here?

The fact is rope-folding puzzles like these can always be solved.

If on each side of a river stand N people each holding k ropes to folk on the other side, then it is possible for each person to drop all but one rope and be left holding a single rope across the river with another person.

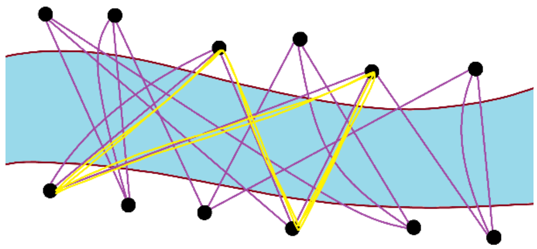
We'll illustrate the proof with the specific example of $N = 6$ and $k = 3$.

Start by selecting a person and a rope she is holding and follow the rope across the river to a person on the opposite side. Follow a different rope this new person is holding back to a new person on the starting side, if possible.* Then follow a different rope that person is holding to a different person, if you can, on the opposite side. And so on, back and forth across the river. As there are only finitely many people, you must eventually have no choice but to return to a person you've visited before. Consider the <u>first</u> time this happens.

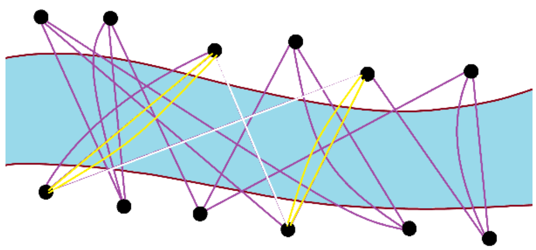


An Annoyance: *What if you reach a person for which <u>every</u> rope that person is holding leads back to the same person whence you came? Can you see this means these two people are holding k ropes between them? In which case, they can each drop all but one rope and they need not be considered in our analysis: study then $N-1$ people each holding k ropes across a river.*

Now, starting with that repeat visitor, we can see that we have a path of ropes back and forth across the river that make a loop. This loop starts and ends on the same side of the river and so there must be an even people holding an even number of ropes in this loop. That number is at least 4.

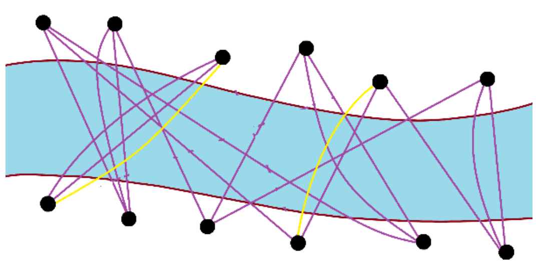


Now have people let go of every second rope in this loop and then pick up the loose ropes to “double up” on the remaining ropes in the loop.

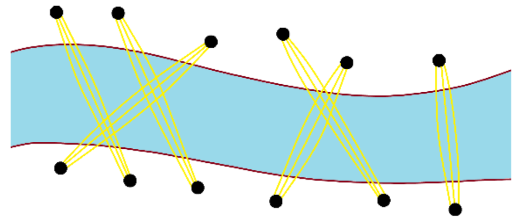


Here we have only removed ropes and added an equal number of ropes between pairs of people who were already holding ropes.

So now we have a new configuration of N people each side of river each holding k ropes.



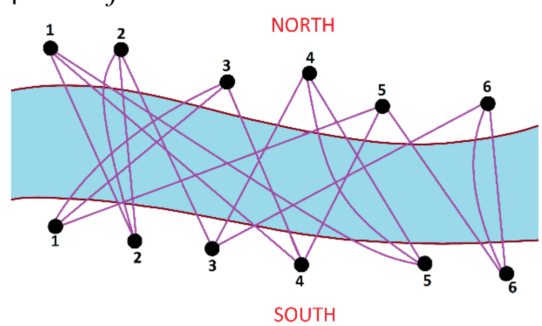
Repeat this process on this new configuration of people and “double up” on even more existing ropes. And do it again and again until there are no loops of 4 or more ropes to be found.



The result is bundles of k ropes between pairs of people who were already holding a rope in the first place. This represents a solution to the rope puzzle!

SEMI-MAGIC SQUARES

Number the people on each side the river 1 through 6 and write in a table the count of ropes person i on one side is holding with person j on the other.



south bank

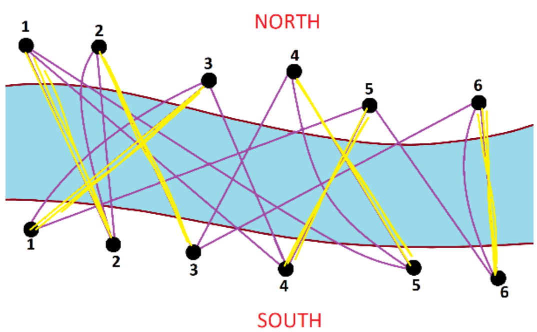
north bank	1	2	3	4	5	6
1	0	1	0	1	1	0
2	0	2	1	0	0	0
3	2	0	0	1	0	0
4	0	0	1	0	2	0
5	1	0	0	1	0	1
6	0	0	1	0	0	2

ESTABLISHING THE FORMULA

This gives a square array of numbers with the property that every row of entries sums to 3 and every column of entries sums to 3 as well. It's what is called a *semi-magic square* with magic sum 3. (To be "fully magic" we'd need the entries in the two main diagonals to sum to 3 as well.)

Every "ropes across a river" scenario gives a semi-magic square with non-negative integer entries, and, conversely, any semi-magic square with non-negative entries can be seen as a "ropes across a river" scenario.

A semi-magic square composed of 0s and 1s with magic sum 1 is called a *permutation matrix.* (It is the identity matrix with its columns—or rows—permuted.) It corresponds to single ropes being held across a river between pairs of people.



south bank

		1	2	3	4	5	6
north bank	**1**	0	1	0	0	0	0
	2	0	0	1	0	0	0
	3	1	0	0	0	0	0
	4	0	0	0	0	1	0
	5	0	0	0	1	0	0
	6	0	0	0	0	0	1

And if we take away these single ropes we're left with a new ropes-across-the-river problem with N people each holding just $k-1$ ropes this time. And we can repeat our analysis yet again. And again! And again.

This proves:

Every semi-magic square of non-negative integers with magic sum k is a sum of k permutation matrices.

$$\begin{bmatrix} 0&1&0&1&1&0\\ 0&2&1&0&0&0\\ 2&0&0&1&0&0\\ 0&0&1&0&2&0\\ 1&0&0&1&0&1\\ 0&0&1&0&0&2 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix} + \begin{bmatrix} 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{bmatrix}$$

THE OPENING PUZZLER

In the first version of the puzzle, we have two piles each of 26 cards with 26 red cards and 26 black cards distributed between them. Any table that gives their distribution must be a 2×2 semi-magic square with magic sum 26.

	#reds	#blacks
pile 1	a	b
pile 2	b	a

a+b = 26

This semi-magic square is a sum of 26 permutation matrices and so the task of puzzle 1 can indeed be accomplished 26 times in a row!

The scenario of puzzle 2 can be encoded via a 4×4 table that is a semi-magic square with magic sum 13, and the scenario of puzzle 3 by a 13×13 table that is a semi-magic square with magic sum 4. Each is a sum of permutation matrices and so each puzzle can be completed, all the way through!

RESEARCH CORNER

Do semi-magic cubes exist? If so, is each semi-magic cube with non-negative integer entries a sum of semi-magic squares of magic sum 1 (“permutation cubes”)?

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