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CURIOUS MATHEMATICS FOR FUN AND JOY

NOVEMBER 2019

THIS MONTHS' PUZZLER:

It's a long one. We're going to present some different sequence constructions and find them extraordinarily inter-related.

Here's a first sequence.

Start by imagining an infinite string of blank spaces.

Place a 0 in the first blank, and then in every second blank thereafter.

Place a 1 in the first available blank, then in every second available blank thereafter.

Place a 2 in the first available blank, then in every second available blank thereafter.

And so on.

0 _ 0 _ 0 _ 0 _ 0 _ 0 _ 0 _ 0 _ 0 _ 0 _ 0 _ 0 _ …
0 1 0 _ 0 1 0 _ 0 1 0 _ 0 1 0 _ 0 1 0 _ 0 1 0 _ …
0 1 0 2 0 1 0 _ 0 1 0 2 0 1 0 _ 0 1 0 2 0 1 0 _ …
0 1 0 2 0 1 0 3 0 1 0 2 0 1 0 _ 0 1 0 2 0 1 0 3 …
0 1 0 2 0 1 0 3 0 1 0 2 0 1 0 4 0 1 0 2 0 1 0 3 …
⋮

This gives the infinite sequence

0 1 0 2 0 1 0 3 0 1 0 2 1 0 1 0 4 0 1 0 2 … .

Here's a second sequence.

Start with 0.

Now perform the following iteration rule: *Given the sequence you have so far, write down that sequence twice, but add* 1 *to the final term.*

0
0 | 1
0 1 | 0 2
0 1 0 2 | 0 1 0 3
0 1 0 2 0 1 0 3 | 0 1 0 2 0 1 0 4
⋮

This gives the infinite sequence
0 1 0 2 0 1 0 3 0 1 0 2 1 0 1 0 4 0 1 0 2 … .

a) Is it obvious that these two different constructions should give the same sequence?

Here's a third sequence.

Write each of the counting numbers in binary.

1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, …

Then list the number 0 s each binary expression ends with. This gives the sequence
0 1 0 2 0 1 0 3 0 1 0 2 1 0 1 0 4 0 1 0 2 … .

b) Why are all three sequences the same?

This persistent sequence is persistent in another way.

Fractal Property: *Delete every appearance of* 0 *in this special sequence. What remains is a copy of the original sequence, but with each entry increased by one.*

~~0~~ 1 ~~0~~ 2 ~~0~~ 1 ~~0~~ 3 ~~0~~ 1 ~~0~~ 2 ~~0~~ 1 ~~0~~ 4 …
= 1 2 1 3 1 2 1 4 …
Subtract 1
= 0 1 0 2 0 1 0 3 …

c) Why is a copy of the sequence hidden within the sequence in this way?

The Summation Sequence
Consider the sequence whose terms are the sum of the first few terms our curious sequence.

0 = 0
0+1 = 1
0+1+0 = 1
0+1+0+2 = 3
0+1+0+2+0 = 3
0+1+0+2+0+1 = 4
⋮

This gives the infinite sequence
0 1 1 3 3 4 4 7 7 8 8 10 10 11 11 …

d) Explain why the N th term of this sequence is the total number of explosions that occur when you place N dots in the rightmost box of a $1 \leftarrow 2$ machine.

[See here for the story of Exploding Dots.]

And for a final piece of craziness …

The number 13 in binary is **1101.**

Delete its final digit to get **110**
Delete the next final digit to get **11**
Delete the next final digit to get **1**
Sum these truncations in binary to get 1010 which is the number ten.

13 = 1101

```
   1 1 0
     1 1
+      1
--------
=  1 2 2 = 1 0 1 0
```

The 13[th] term of the sequence
0 1 1 3 3 4 4 7 7 8 8 10 10 11 11 …
is ten!

The 50[th] term of the sequence is 48.

50 = 110010

```
  1 1 0 0 1
    1 1 0 1
      1 1 0
        1 1
+         1
-----------
= 1 2 2 2 4 = 1 1 0 0 0 0
```

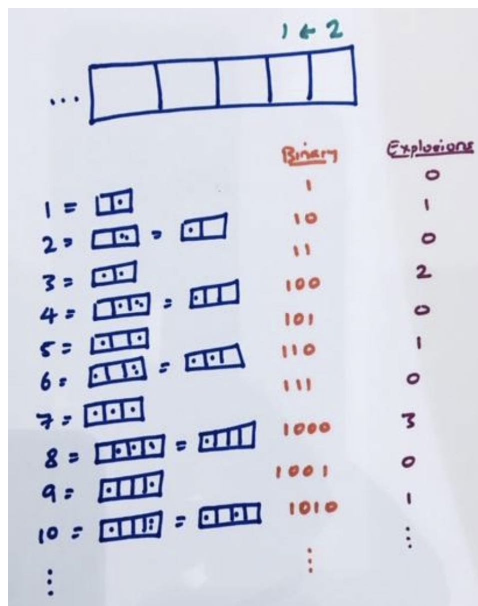
11000 *is forty-eight in binary*.

e) What is going on? Why is the total number of explosions that occur when you place N dots in the rightmost box of a $1 \leftarrow 2$ machine given by this bizarre truncate-and-sum procedure?

COUNTING EXPLOSIONS IN A $1 \leftarrow 2$ MACHINE ONE DOT AT A TIME

In the story of EXPLODING DOTS a $1 \leftarrow 2$ machine is simply a row of boxes extending as far to the left as one pleases. One places dots into this machine, always in the rightmost box. But whenever there are two dots in a box they disappear—EXPLODE!—to be replaced by one dot in the box one place to their left.

Putting in dots one at a time shows that this machine produces the binary codes of numbers.



The Order in Which One Conducts Explosions Does Not Matter.
In this picture we inserted dots one at a time in the rightmost box and conducted explosions. Once could also place dots into the rightmost box all at once to work out the binary code of a particular number and conduct explosions in a haphazard manner: *All sequences of explosions lead to the same final binary code.* After a moment's thought, you come to realise that this is actually surprising!

Here's why order does not matter.

With N dots in the rightmost box, the number of explosions that occur in that box is fixed (its half of N if N is even, half of $N-1$ if N is odd). Even if you delay conducting some of them, the count of explosions fixed.

Each explosion in the rightmost box produces a dot in the second box, the one to its left. This means that the count of dots that ever appear there is fixed, and so too then is the count of explosions that occur in that second box, even if you choose to delay conducting some of those explosions.

This means the count of dots that ever appear in the third-to-left box is fixed, and so is the count of explosions that occur there.

And so on.

We see that the count of explosions that occur in each box is fixed at the outset by the number N of dots that are placed in the first box. Thus the final code that appears for N in a $1 \leftarrow 2$ machine is fixed too!

Challenge: Prove that "order does not matter" also for a $2 \leftarrow 3$ machine.

Let's count explosions.

Set $E(N)$ to be the count of explosions that occur with the introduction of an N th dot, after all possible explosions that could occur with $N-1$ dots placed in the machine have taken place.

If $N-1$ is even, then its final digit in binary is 0. Also, an even number of dots placed into a $1 \leftarrow 2$ explode to leave the rightmost box empty. Adding an N th dot into the machine after $N-1$ dots have settled thus causes no new explosions—the one new dot simply sits in the final box of the machine. So $E(N)=0$. We also see that the binary code of N ends with a 1.

$E(odd)=0$.

If $N-1$ is odd, then its binary code ends with a string of 1s. Suppose it ends with k of them, that is, $N-1$ appears in binary as $....0\overbrace{11...1}^{k}$. This tells us that when $N-1$ dots placed in a $1 \leftarrow 2$ machine they settle after all explosions to an arrangement ending with a string of single dots in the last k boxes (and an empty box just to their left). Adding an N th dot into the machine then causes k explosions. So $E(N)=k$. But we also see that the resulting binary code for N ends with k zeros—it's of the form $...1\overbrace{00...0}^{k}$.

Either way, for N odd or even, we see:

$E(N)=$ the count of trailing zeros in the binary code for N.

Thus $\{E(N)\}$ gives the third sequence
0 1 0 2 0 1 0 3 0 1 0 2 1 0 1 0 4 0 1 0 2 …
of the opening puzzler.

In binary, multiplying a number by two simply addends a 0 to the binary code of the number. Thus $2N$ has one more trailing zero than N does. We have

$E(2N)=E(N)+1$.

Since the only numbers for which $E(N)$ equals zero is N odd, this reads:

Deleting all the 0 s from the sequence $\{E(N)\}$ leaves entries one more than the original sequence.

Since the 0s of the original sequence are in every odd position of the original sequence this means all the 1s in the original sequence, one more than the 0s, are in every odd position once the 0s are removed.

The 2s in the original sequence (one more than the 1s) are "behaving like" the 1s once the 0s are removed. And so they are in every odd position of the sequence once the 1s are removed as well.

The 3s in the original sequence (one more than the 2s) are behaving like the 2s once the 0s are removed. And so are in every odd position of the original sequence once the 0s and 1s and 2s are removed.

And so on.

This matches precisely the construction given for the first sequence in the opening puzzler.

The sequence $\{E(N)\}$ can be constructed by the manner of the first sequence in the opening puzzler.

The sequence given by the second construction of the opening puzzler has us look at elements in positions $1,2,,...2^k$ and in positions $2^k+1,\ 2^k+2,...,2^k+2^k$, for each value of k. If the sequence $\{E(N)\}$ is to follow the construction suggested there, we need to prove three things:

1. $E(1)=0$,
2. $E(2^k+a)=E(2^k)$ for $1\le a<2^k$,
3. $E(2^k+2^k)=E(2^k)+1$.

Item 1 is clear.

Item 3 follows from $E(2N)=E(N)+1$.

Item 2 is the interesting one.

If $1\le a<2^k$, then, in binary, a is at most k digits long. With leading zeros, let's say it is k digits long. Then 2^k+a has the same binary code as a but has a digit 1 added to its front. Thus a and 2^k+a have the same count of trailing zeros and so property 2 follows.

The first three sequences outlined in the opening puzzler really are the same sequence. And the Fractal Property of this sequence, as described in the puzzler, holds.

COUNTING EXPLOSIONS IN A $1\leftarrow2$ MACHINE, ALL DOTS IN ONE HIT

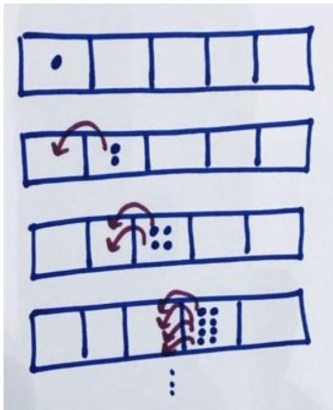
Let $T(N)$ be the total count of explosions that occur when you place N dots directly in the rightmost box of a $1\leftarrow2$ machine.

We saw earlier that this number is well-defined: the total count of explosions that occur is independent of the order one chooses to conduct them. So we might as well place the dots in one at a time and count the explosions that occur each time. Thus $T(N)=E(1)+E(2)+\cdots+E(N)$ and the sequence $\{T(N)\}$ is the "summation sequence" given by the latter part of the opening puzzler.

Let's understand the count $T(N)$ another way.

Consider a dot sitting anywhere in a $1\leftarrow2$ machine and let's ask: how many explosions must have occurred to get that dot there?

Well, a dot in any one position (except the very last) must have come from two dots exploding in the box to its right. That's 1 explosion. But two dots in that box must have come from 2 explosions, namely, from four dots in the box to their right exploding. And the four dots there must have come from 4 explosions from eight dots the box to its right. And so on. A dot in a box k steps to the left of the rightmost box must be the result of $1+2+4+\cdots+2^k$ explosions.



But in binary the number $1+2+4+\cdots+2^k$ reads as $111\cdots1$, and this is the result of the following binary sum.

```
  1000...0
   100...0
    10...0
         ⋮
+        1
----------
= 1111... 1
```

Actually, this is true of each and every dot that represents a 1 in the binary code of a given number N : we can imagine repeatedly sliding each 1 one place to the right (which is accomplished by repeatedly truncating the binary code of N) and computing a binary sum.

50 = 110010

```
  11001
   1101
    110
     11
+     1
-------
= 12224 = 110000
```

11000 is forty-eight in binary.

We have $T(50)=48$.

Thus $T(N)$ is the result of that binary sum.

We have now made sense of all the pieces of the puzzler! Whoa!

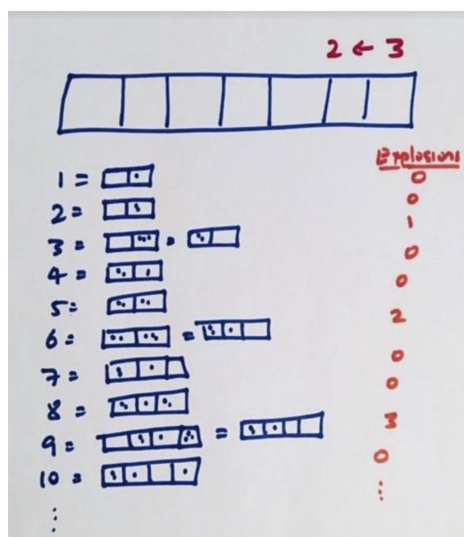
RESEARCH CORNER

Consider a $2 \leftarrow 3$ machine.

Let's count the explosions that occur as we place dots in this machine one at a time (completing all the explosions that can possibly occur as we go along).

This gives the sequence

0 0 1 0 0 2 0 0 3 0 0 1 0 0 4 0 0 2 0 0 1 0 0 5 0 0 3 0 0 1 0 0 2 0 0 6 0 0 1 0 0 4 0 0 2 0 0…



Denote the N th term of this sequence as $F(N)$.

As explosions occur only with the placement of every third dot, we have

$F(N) = 0$ if N is not a multiple of three.

(just like $E(N) = 0$ if N is not a multiple two when thinking of a $1 \leftarrow 2$ machine).

If we focus on the non-zero terms we get the sequence

1 2 3 1 4 2 1 5 3 1 2 6 1 4 2 1 3 7 1 2 5 …

This sequence is fascinating.

a) Prove the following

$$F(9k) = F(6k) + 1$$
$$F(9k+3) = 1$$
$$F(9k+6) = F(6k+3) + 1$$

for $k = 0, 1, 2, 3, \ldots$.

b) Fractal Property: Prove that if one deletes all the 1s from this sequence what remains is a copy of the original sequence with all entries raised by one.

c) Explain why this sequence can also be constructed as follows:

Imagine an infinite line of blanks.

Place a 1 *in the first blank and then every third blank thereafter.*

Place a 2 *in the first available blank, and then every third available blank thereafter.*

Place a 3 *in the first available blank, and then every third available blank thereafter.*

And so on.

d) **UNKNOWN TO ME:** Is there another way to construct this sequence, analogous to the second approach in the opening puzzler? (Do we need to put the 0 s back in to see it?)

MORE MACHINES

We've played with $1 \leftarrow 2$ and $2 \leftarrow 3$ machines. Consider then a general $b-1 \leftarrow b$ machine for some integer $b > 1$.

Let $G(N)$ be the count of explosions that occur when one places an N th dot in the machine (assuming all the explosions with $N-1$ dots before it have occurred).

We have

$G(N) = 0$ if N is not a multiple of b.

Let's focus on the non-zero terms of the sequence.

Am I right to believe the following?

$G(kb) = 1$ if $k-1$ is a multiple of b.

$$G(kb) = G\left(\left(k - \left\lceil \frac{k-1}{b} \right\rceil\right)b\right) + 1$$

if $k-1$ is not a multiple of b.

What curious of properties of the sequence of non-zero values do these recursion relations imply?

EVEN MORE MACHINES

Count explosions in $1 \leftarrow 3$ and $2 \leftarrow 4$ and $3 \leftarrow 5$ machines, and so on.

What structures in sequences are to be found?

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